Bachelor-Thesis

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Static Axially Symmetric Deformation of the Schwarzschild Spacetime

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Abstract

We investigate a static, axisymmetric solution of the Einstein field equations. As a special solution of the Weyl class the Erez-Rosen spacetime extends the Schwarzschild spacetime by a quadrupole moment. We present a physical interpretation for this spacetime and a classification of equatorial timelike geodesics sufficiently far away from the gravitational source.

Furthermore, we make a first step toward an exploration of closed, non-planar geodesics. Some examples are obtained by deformation of circular equatorial orbits.

Contents

1 Introduction

400 years ago Johannes Kepler found his famous laws of planetary motion. According to these laws the orbit of a planet is an ellipse with the sun in one of its foci. Kepler found these laws by analyzing the empirical data observed by the astronomer Tycho Brahe. The laws had no deeper theoretical foundation, but they were found to describe the planetary orbits very accurately.

In 1687, Isaac Newton developed in his *Philosophiae Naturalis Principia Mathematica* a theory of gravity that implied all three Kepler's laws, thus giving them a satisfactory basis. But after having taken the perturbations due to the other planets into account, a small deviation between the prediction of Newton's theory and astronomical observations remained. In November 1915 Albert Einstein resolved this problem in his theory of General Relativity.

Only a month later the German physicist Karl Schwarzschild presented the first exact solution of Einstein's equations describing a spacetime around a spherical mass. By investigating the geodesic motion of test particles in this spacetime, Kepler's laws were improved and ellipses turned into rosettes.

Typical astronomical objects are not perfectly spherical mostly due to their rotation, which causes a flattening of the object. This effect is extremely small for our sun which is indeed nearly a perfect sphere. The reason for this is the relatively small angular velocity and the sun's enormous mass. The flattening is stronger for fast-rotating stars or planets like the earth. A *deformation of the Schwarzschild spacetime* is required with a deviation from spherical symmetry. For an oblate spheroidal mass distribution the solution of the vacuum Einstein equations has to be *axially symmetric* and *static* if we disregard rotation.

Such a static, axisymmetric spacetime is described by the so-called Weyl metric, due to Hermann Weyl and Tullio Levi-Civita. The Einstein vacuum field equations then turn out to be quite simple and can be solved analytically. The resulting family of solutions can be interpreted as deformations of the Schwarzschild solution taking arbitrary multipole moments into account. The simplest member is named after Nathan Rosen and G. Erez. It extends the Schwarzschild solution by a quadrupole moment q.

We recall the derivation of this solution as a special solution of the Weyl class and try to interpret it physically by considering the Newtonian limit (and also briefly covariant multipole moments). Furthermore, we investigate geodesics in this spacetime with an emphasis on equatorial orbits and compute the tiny contribution to the perihelion shift of Mercury's orbit due to the sun's quadrupole moment. Finally we explore some non-planar geodesics and obtain examples of closed non-planar orbits.

Throughout this thesis we use natural units, hence $G = c = 1$. The results given in this thesis have been checked by using the computer algebra system MATHEMATICA.

2 Static axially symmetric spacetimes - The Weyl metric

In the case of a static, axially symmetric spacetime, the line element can be chosen as

$$\mathrm{d}s^2 = e^{2\psi}\mathrm{d}t^2 - e^{2(\gamma-\psi)}\left(\mathrm{d}\rho^2 + \mathrm{d}z^2\right) - \rho^2 e^{-2\psi}\mathrm{d}\phi^2\,, \tag{1}$$

where ψ and γ are functions of the coordinates ρ, z. This was shown by Levi-Civita [14] and Weyl [29]. We call (t, ρ, z, ϕ) cylindrical Weyl coordinates.

2.1 Special case: The Schwarzschild solution

Of course the line element (1) has to contain that of the Schwarzschild solution. Indeed, we obtain the well-known Schwarzschild line element

$$\mathrm{d}s^2 = \left(1 - \frac{2M}{r}\right)\mathrm{d}t^2 - \frac{1}{\left(1-\frac{2M}{r}\right)}\mathrm{d}r^2 - r^2\mathrm{d}\theta^2 - r^2\sin^2\theta\,\mathrm{d}\phi^2$$

by using the coordinate transformation

$$\rho = \sqrt{r^2 - 2Mr}\,\sin\theta\,, \quad z = (r-M)\cos\theta$$

and choosing

$$\psi = \frac{1}{2}\ln\left[1 - 2M/r\right], \quad \gamma = \frac{1}{2}\ln\left[\frac{(1-2M/r)}{(1-M/r)^2}\right]. \tag{2}$$

2.2 Field equations and their solutions

As derived in appendix A[1], the Einstein equations in vacuum are given by

$$\psi_{\rho\rho} + \frac{1}{\rho}\psi_\rho + \psi_{zz} = 0\,, \tag{3}$$

$$\gamma_\rho = \rho\left(\psi_\rho^2 - \psi_z^2\right), \quad \gamma_z = 2\rho\psi_\rho\psi_z\,. \tag{4}$$

The equation (3), which determines ψ, is the three dimensional Laplace equation in cylindrical coordinates, where the ϕ-term is missing due to the axial symmetry. In spherical coordinates (r, θ, φ) it becomes

$$\psi_{rr} + \frac{1}{r^2}\psi_{\theta\theta} + \frac{2}{r}\psi_r + \frac{1}{r^2}\cot\theta\,\psi_\theta = 0$$

which is solved by [26]

$$\psi = \sum_{n=-\infty}^{\infty} A_n r^n P_n(\cos\theta)\,. \tag{5}$$

[1] In appendix A we allow the functions ψ and γ to be time-dependent. The static equations we are interested in here are presented in A.1.

P_n are the Legendre polynomials which satisfy $P_{-n-1} = P_n$.

If the coefficients A_n are given by

$$A_k = 0 \text{ for } k \geq 0\,, \quad A_{-1} = -M\,, \quad A_{-2} = 0\,, \quad A_{-3} = -\frac{1}{3}M^3\,, \quad \text{etc.}$$

the series (5) converges to the Schwarzschild-solution [20].

It will be convenient to use ellipsoidal coordinates (λ, μ) [4]. These are connected with the Weyl coordinates via

$$\lambda = \frac{1}{2M}\left(\sqrt{\rho^2 + (z+M)^2} + \sqrt{\rho^2 + (z-M)^2}\right),$$
$$\mu = \frac{1}{2M}\left(\sqrt{\rho^2 + (z+M)^2} - \sqrt{\rho^2 + (z-M)^2}\right),$$

and the inverse relation

$$\rho = M\sqrt{\lambda^2 - 1}\sqrt{1-\mu^2}\,, \quad z = M\lambda\mu\,.$$

Therefore

$$\lambda \geq 1\,, \quad -1 \leq \mu \leq +1$$

applies. The parameter M is associated with the mass that generates the gravitational field. In these coordinates, (3) becomes

$$\frac{\partial}{\partial\lambda}\left[(1-\lambda^2)\psi_\lambda\right] = \frac{\partial}{\partial\mu}\left[(1-\mu^2)\psi_\mu\right]. \tag{6}$$

By using a separation ansatz $\psi = f(\mu)g(\lambda)$ we obtain

$$\frac{\mathrm{d}}{\mathrm{d}\mu}\Big[(1-\mu^2)f'(\mu)\Big] + af(\mu) = 0 \quad \text{and} \quad \frac{\mathrm{d}}{\mathrm{d}\lambda}\Big[(1-\lambda^2)g'(\lambda)\Big] + ag(\lambda) = 0\,. \tag{7}$$

These equations are known as Legendre's differential equation for f and g respectively. In order to avoid singularities of $f(\mu)$ at $\mu = \pm 1$, we demand that the separation constant a be an integer [21] and choose it to be $l(l+1), l \in \mathbb{Z}$. Thus the general solution is

$$f(\mu) = aP_l(\mu) + bQ_l(\mu)\,, \quad g(\lambda) = cP_l(\lambda) + dQ_l(\lambda)\,,$$

where P_l are the Legendre-polynomials, Q_l are the Legendre-functions of the second kind (see appendix B) and a, b, c, d are real coefficients.

Since we are only interested in asymptotically flat metrics, the functions f and g have to vanish in the limits $z \to \infty$ or $\rho \to \infty$. For $z, \rho \to \infty$ the coordinate λ diverges and so does $P_l(\lambda)$. Consequently c must be zero. The coordinate μ tends to 0 for $\rho \to \infty$ and to 1 for $z \to \infty$, and $Q_l(\mu)$ then diverges. Therefore b must also be zero. We obtain

$$\psi_l = q_l\, P_l(\mu)Q_l(\lambda) \quad \text{with} \quad q_l = ad\,, \tag{8}$$

and more generally a superposition

$$\psi = \sum_{l=0}^{\infty} q_l\psi_l \quad \text{with} \quad \psi_l := P_l(\mu)\, Q_l(\lambda)\,, \tag{9}$$

where q_l are real coefficients.

2.3 Deformation of the Schwarzschild solution

The axially symmetric spacetimes determined by (9), in ellipsoidal coordinates, contain the Schwarzschild-solution. We recover it from

$$\psi_S := -\psi_0 = -P_0(\mu)\, Q_0(\lambda) = \frac{1}{2} \ln \frac{\lambda - 1}{\lambda + 1}\,,$$

for which

$$\gamma_S = \frac{1}{2} \ln \Big(\frac{\lambda^2 - 1}{\lambda^2 - \mu^2} \Big)$$

results from the equations (4).

Now we introduce Schwarzschild coordinates $(\bar{r}, \bar{\theta})$, which should not be mixed up with the spherical coordinates used in 2.2. Via the coordinate transformation $(\bar{r}, \bar{\theta}) \to (\lambda, \mu)$ given by

$$\lambda = \frac{\bar{r}}{M} - 1\,, \quad \mu = \cos \bar{\theta}\,,$$

we recover the familiar Schwarzschild line element,

$$\mathrm{d}s^2 = \Big(1 - \frac{2M}{\bar{r}}\Big)\mathrm{d}t^2 + \Big(1 - \frac{2M}{\bar{r}}\Big)^{-1} \mathrm{d}\bar{r}^2 + \bar{r}^2 \mathrm{d}\bar{\theta}^2 + \bar{r}^2 \sin^2 \bar{\theta}\, \mathrm{d}\phi^2\,.$$

We rewrite (9) in the form

$$\psi = \psi_S + \sum_{l=0}^{\infty} q_l \psi_l = \frac{1}{2} \ln \Big(\frac{\lambda - 1}{\lambda + 1} \Big) + \sum_{l=1}^{\infty} q_l P_l(\mu)\, Q_l(\lambda)\,. \tag{10}$$

This will be interpreted as the Schwarzschild solution generalized by terms with multipole moments $q_l, l > 0$, though further discussions concerning the notion of multipole moments in general relativity are required, see 3.2.

In view of the systems we want to describe, e.g. flattened stars, we demand the existence of a reflection symmetry with the symmetry plane given by $\mu = 0$, thus the equatorial plane. Therefore $q_l = 0$ for all odd l, because $P_{2l+1}(\mu)$ changes its sign under the symmetry transformation $\mu \to -\mu$.

3 The Erez-Rosen spacetime

The Erez-Rosen solution is of the form (10), where beyond Schwarzschild only a quadrupole term is taken into account, hence $q_l = 0$ for all $l > 2$ ($q_2 \equiv q, \psi_R \equiv -\psi_2$).

$$\begin{aligned} &\psi = \psi_S + q\psi_R \\ \text{with} \quad &\psi_R := \frac{1}{2}(3\mu^2 - 1)\left[\frac{1}{4}(3\lambda^2 - 1)\ln\left(\frac{\lambda - 1}{\lambda + 1}\right) + \frac{3}{2}\lambda\right]. \end{aligned}$$

Substituting this into the field equations for γ and integrating the equations one obtains [2]

$$\gamma = \gamma_S + q\gamma_R \,,$$

with

$$\begin{aligned} \gamma_R := &\ln\left(\frac{\lambda^2 - 1}{\lambda^2 - \mu^2}\right) + (\mu^2 - 1)\left[\frac{3}{2}\lambda\ln\left(\frac{\lambda - 1}{\lambda + 1}\right) + 3\right] \\ &+ q\Bigg[\frac{1}{2}\ln\left(\frac{\lambda^2 - 1}{\lambda^2 - \mu^2}\right) + (1 - \mu^2)\Big[\frac{9}{64}(\lambda^4 - 2\lambda^2 + 1)\ln^2\left(\frac{\lambda - 1}{\lambda + 1}\right) \\ &+ \frac{3}{16}(3\lambda^3 - 5\lambda)\ln\left(\frac{\lambda - 1}{\lambda + 1}\right) + \frac{3}{16}(3\lambda^2 - 4)\Big] + \mu^2(\mu^2 - 1) \\ &\Big[\frac{9}{64}(9\lambda^4 - 10\lambda^2 + 1)\ln^2\left(\frac{\lambda - 1}{\lambda + 1}\right) + \frac{9}{16}(9\lambda^3 - 7\lambda)\ln\left(\frac{\lambda - 1}{\lambda + 1}\right) + \frac{9}{16}(9\lambda^2 - 4)\Big]\Bigg]. \end{aligned}$$

In Schwarzschild coordinates, where the bars are now omitted, the Erez-Rosen line element takes the form

$$\begin{aligned} \mathrm{d}s^2 = &e^{2\psi}\mathrm{d}t^2 - e^{2(\gamma - \psi)}\left[\left(1 + \frac{m^2\sin^2\theta}{r^2 - 2mr}\right)\mathrm{d}r^2 + \left(r^2 - 2mr + m^2\sin^2\theta\right)\mathrm{d}\theta^2\right] \\ &- e^{-2\psi}(r^2 - 2mr)\sin^2\theta\,\mathrm{d}\phi^2 \,. \end{aligned}$$

The r- and θ- depending functions ψ and λ are given [24] by[3]

$$\begin{aligned} &\psi = \psi_S + q\,\psi_R\,, \quad \gamma = \gamma_S + q\,\gamma_R \quad \text{with} \\ &\psi_S = \frac{1}{2}\ln\left(1 - \frac{2m}{r}\right), \qquad (11) \\ &\psi_R = \frac{1}{4}(3\cos^2\theta - 1)\left[\frac{1}{2m^2}(3r^2 - 6mr + 2m^2)\ln\left(1 - \frac{2m}{r}\right) + 3\frac{r - m}{m}\right], \qquad (12) \\ &\qquad (13) \end{aligned}$$

[2]There is a misprint in [4], which is corrected for example in [32] and [33].
[3]There is a misprint in [24]. In the second term of the expression for γ_R there is a factor $\sin^2\theta$ missing.

$$\gamma_S = \frac{1}{2}\ln\left(\frac{r^2-2mr}{r^2-2mr+m^2\sin^2\theta}\right), \tag{14}$$

$$\begin{aligned}\gamma_R &= \frac{1}{2}(2+q)\ln\left(\frac{r^2-2mr}{r^2-2mr+m^2\sin^2\theta}\right) - \frac{3}{2m}\left[(r-m)\ln\left(1-\frac{2m}{r}\right)+2m\right]\sin^2\theta \\ &+ \frac{9}{16}q\sin^2\theta\left\{\left(\frac{r-m}{m}\right)^2(1-9\cos^2\theta)+4\cos^2\theta-\frac{4}{3}\right. \\ &+ \frac{r-m}{m}\left[\left(\frac{r-m}{m}\right)^2(1-9\cos^2\theta)+7\cos^2\theta-\frac{5}{3}\right]\ln\left(1-\frac{2m}{r}\right) \\ &+ \left.\frac{1}{4}\frac{r^2-2mr}{m^2}\left[\left(\frac{r-m}{m}\right)^2(1-9\cos^2\theta)-\sin^2\theta\right]\ln^2\left(1-\frac{2m}{r}\right)\right\}.\end{aligned} \tag{15}$$

$q=0$ yields the Schwarzschild spacetime.

3.1 Quadrupole moment and the Erez-Rosen parameter q

It is natural to associate the parameter q with a quadrupole moment. We substantiate this by comparing the Erez-Rosen solution in the Newtonian limit with a result of the Newtonian theory of gravity.

3.1.1 Multipole moments in Newton's theory of gravity

The Newtonian gravitational potential $\Phi(\vec{r})$ of a mass density $\rho(\vec{r})$ is obtained as the asymptotically vanishing solution of the Poisson equation

$$\Delta\Phi(\vec{r}) = 4\pi G\rho(\vec{r}).$$

In vacuum ($\rho(\vec{r})=0$) this is the Laplace equation. Assuming that the mass distribution is axially symmetric, the solution of the Laplace equation is given by [1]

$$\Phi(r,\theta) = \underbrace{-\frac{M}{r}}_{\text{spherical part}} + \underbrace{\sum_{k=1}^{\infty} Q_k \frac{P_k(\cos\theta)}{r^{k+1}}}_{\text{deviation from spherical symmetry}}.$$

The potential's deviation from spherical symmetry is expressed through the coefficients Q_k of the series, which are called multipole moments. The mass $M=Q_0$ is the monopole moment.

The system we want to describe has an additional plane of symmetry. The sun, for example, can be approximated by an oblate spheroid. The plane of symmetry is determined by $\theta=\frac{\pi}{2}$. Hence the terms with an odd k have to vanish and we obtain

$$\Phi_{\text{Newton}}(r) := -\frac{M}{r} + \frac{Q}{2}\frac{1}{r^3} + \mathcal{O}(r^{-5}), \tag{16}$$

where only the monopole moment M and the quadrupole moment $Q\equiv Q_2$ are considered.

3.1.2 The Newtonian potential of a spheroid

An ellipsoid is a geometric body whose surface is described by the equation

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1\,,$$

with real, positive constants a, b and c. If $a = b \neq c$, we call the ellipsoid a spheroid.

A spheroid is thus determined by its equatorial radius a and its polar radius c. We call it "oblate" if $a > c$ and "prolate" if $a < c$, both cases are visualized in Fig. 1. The Newtonian potential of an oblate, homogenous mass distribution

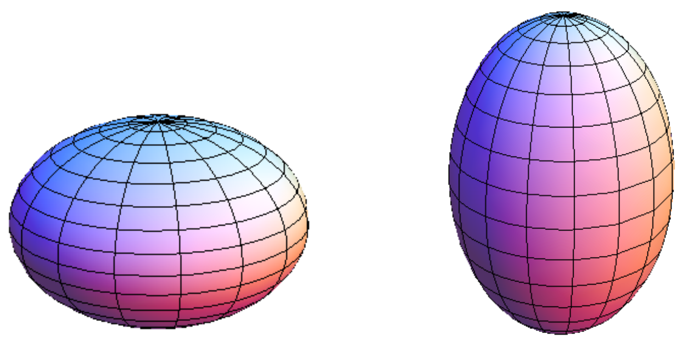


Figure 1: Oblate and prolate spheroids.

with mass density ρ is given [16, p. 62] by

$$\begin{aligned}\Phi_{\text{oblate}}(x,y,z) = &-\frac{2\pi\rho a^2 c}{\sqrt{a^2-c^2}}\left(1-\frac{x^2+y^2-2z^2}{2(a^2-c^2)}\right)\arcsin\sqrt{\frac{a^2-c^2}{a^2+\kappa}}\\ &-\frac{\pi\rho a^2 c\,(x^2+y^2)\sqrt{c^2+\kappa}}{(a^2-c^2)(a^2+\kappa)}+\frac{2z^2\pi\rho a^2 c}{(a^2-c^2)\sqrt{c^2+\kappa}}\,,\end{aligned}$$

where κ is defined by

$$\frac{x^2+y^2}{a^2+\kappa} + \frac{z^2}{c^2+\kappa} = 1\,.$$

In order to identify what corresponds to the quadrupole moment Q, we look at the plane of symmetry ($z = 0$), transform the potential into spherical coordinates and look at the leading terms of the Taylor series:

$$\Phi_{oblate}(r) = -\frac{M}{r} - \frac{a^2-c^2}{10}\frac{M}{r^3} - \frac{9(a^2-c^2)^2}{280}\frac{M}{r^5} - \frac{5(a^2-c^2)^3}{336}\frac{M}{r^7} + \mathcal{O}\Big(\frac{1}{r^8}\Big)\,, \tag{17}$$

where $M = \frac{4}{3}\pi a^2 c\rho$ is the total mass of the oblate mass distribution. It is obvious that for $a = c$ we get the spherical part of (16). By comparing the second terms of (16) and (17), we get an expression for Q in terms of the equatorial and polar radii a and c,

$$Q = -\frac{a^2 - c^2}{5} M \,, \tag{18}$$

which, as expected, vanishes in the case of spherical symmetry.

3.1.3 Relation to the Erez-Rosen parameter

In the Newtonian limit one obtains a relation between the metric and the Newtonian potential,

$$g_{00} = 1 + 2\Phi_{Newton} \,.$$

We expand g_{00} and obtain

$$e^{2(\psi_S + q\psi_R)} = 1 - \frac{2M}{r} + \frac{2q}{15}\frac{M^3}{r^3} + \mathcal{O}\Big(\frac{1}{r^4}\Big) \,.$$

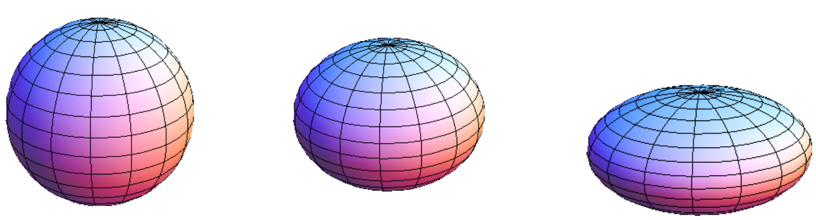
This can be compared to (16). We get a relation between the parameter q and the Newtonian quadrupole moment Q,

$$q = \frac{15Q}{2M^3} \,. \tag{19}$$

With (18) we can derive a relation between an oblate mass distribution and q,

$$q = -\frac{3}{2M^2}\big(a^2 - c^2\big) \,.$$

Some examples of oblate spheroids and their Erez-Rosen parameter q can be found in Fig. 2.



(a) $c = 1, q = 0$ (b) $c = 0.75, q = -0.656$ (c) $c = 0.5, q = -1.125$

Figure 2: Oblate spheroids, and the corresponding value of q ($a = 1, M = 1$).

3.1.4 The quadrupole moment and Erez-Rosen parameter of the sun

We use the measured value for the quadrupole J_2 of the sun in order to calculate $q_\odot$. Instead of Q, we use the dimensionless

$$J_2 = -\frac{Q}{R^2 M},$$

where R is the average radius of the considered object. In case of the sun, new measurements (see for example [19]) indicate a value of

$$J_{2\odot} \approx 2 \cdot 10^{-7}.$$

Consequently we obtain a value for q by using (19) and the sun's properties [30],

$$q_\odot = -\frac{15}{2}\frac{R_\odot^2}{M_\odot^2} J_{2\odot} \approx -3 \cdot 10^5. \tag{20}$$

3.2 Covariant multipole moments

We can try to define multipole moments naively in terms of a Euclidean radial coordinate, assuming that the spacetime differs little from the Euclidean. We expand the metric component g_{00} in terms of this radial coordinate and compare the result with a Newtonian potential. That is exactly what we did in 3.1.3. However, this heuristic definition of multipole moments is coordinate dependent as shown in [21]. An invariant characterization in terms of the geometry of the physical spacetime is lacking.

Geroch found a coordinate independent definition of multipole moments for static spacetimes [6, 7], which was generalized to stationary cases by Hansen [9][4]. In a static spacetime a timelike Killing vector field exists. The orbits of this Killing vector field constitute a three-dimensional space. By using an additional spacelike Killing vector field, that typically exists in an axisymmetric case, we are able to construct invariants of this 3-geometry.

With these covariant multipole moments we are able to describe a given solution of the Einstein field equations or to compare two similar spacetimes.

The first seven Geroch-Hansen multipole moments M_i^{ER} of the Erez-Rosen spacetime are given by [21]

$$\begin{aligned}
&M_0^{ER} = m\,, \quad M_1^{ER} = 0\,,\\
&M_2^{ER} = \frac{2}{15} q m^3\,, \quad M_3^{ER} = 0\,,\\
&M_4^{ER} = -\frac{4}{105} q m^5\,, \quad M_5^{ER} = 0\,,\\
&M_6^{ER} = \frac{2}{15}\frac{4}{231} q m^7 \left(\frac{194}{7} + \frac{14}{15} q\right) + \frac{2}{7}\frac{817}{33} m^2 M_4^{ER}\,, \quad M_7^{ER} = 0\,.
\end{aligned}$$

[4]Other covariant definitions were presented by Thorne or Beig and Simon [21]. The Beig-Simon definition leads to the same multipole moments as the Geroch-Hansen definition. The Thorne multipole moments differ by a constant factor only.

With $q = 0$ we obtain the result for the Schwarzschild solution, all multipole moments higher than the monopole vanish, which should always be the case for a reasonable definition of multipole moments. The odd Geroch-Hansen (GH) multipole moments of the Erez-Rosen spacetime also vanish as expected. Furthermore, we see that the relativistic quadrupole moment M_2^{ER} coincides with the Newtonian quadrupole moment Q in (19). We obtain that the Erez-Rosen spacetime has higher GH-multipole moments beyond the quadrupole moment, but multipole moments M_l^{ER} with $l > 2$ vanish in the Newtonian limit $c \to \infty$.

By superposing solutions from the general Erez-Rosen class (8) in the following way,

$$\psi_{(M-Q)} := \sum_{\alpha=0}^{\infty} q^{\alpha} \psi_{\alpha} \quad \text{with} \quad \psi_{\alpha} = P_{\alpha}(\mu)\, Q_{\alpha}(\lambda)\,,$$

one can generate a solution that only possesses a GH-monopole and a GH-quadrupole moment. This solution is called $(M-Q)$ solution [12]. Because of the linearity of the field equations we can cut off this series at any order α and obtain again an exact solution. This solution possesses a GH-monopole and a GH-quadrupole moment, further multipole moments vanish up to order $2(\alpha+1)$ (included). It represents a spacetime with a small GH-quadrupolar correction to the Schwarzschild solution, because higher GH-multipole moments are proportional to higher powers of q. Further thoughts on this solution follow in section 5.

4 Geodesics in the Erez-Rosen spacetime

First we will derive the geodesic equation. An exploration of geodesics in the equatorial plane is presented following [24] [5]. Finally we consider non-planar geodesics in the Erez-Rosen spacetime.

4.1 The geodesic equations

From the Lagrangian

$$\begin{aligned}\mathscr{L} &:= g_{\mu\nu}\dot{x^{\mu}}\dot{x^{\nu}} \\ &= \left(1-\frac{2m}{r}\right)e^{2q\psi_R}\dot{t}^2 - e^{2q(\gamma_R-\psi_R)}\left[\frac{1}{1-2m/r}\dot{r}^2 + r^2\dot{\theta}^2\right] - r^2\sin^2\theta\, e^{-2q\psi_R}\dot{\phi}^2\,,\end{aligned}$$

we read off two constants of motion, which represent the test particle's energy and angular momentum, respectively:

$$E := \left(1-\frac{2m}{r}\right)e^{2q\psi_R}\dot{t} = \text{Const.}\,, \tag{21}$$

$$L := r^2\sin^2\theta\, e^{-2q\psi_R}\dot{\phi} = \text{Const.}\,. \tag{22}$$

[5] Other explorations of geodesics in the Erez-Rosen spacetime were presented in [1, 13] for example. A compendium of further relating publications was presented in [25].

The two remaining Euler-Lagrange equations are

$$\frac{\mathrm{d}}{\mathrm{d}s}\Big[-2\frac{e^{2q(\gamma_R-\psi_R)}}{1-2m/r}\dot{r}\Big] = \frac{\partial\mathscr{L}}{\partial r}\,, \tag{23}$$

$$\frac{\mathrm{d}}{\mathrm{d}s}\Big[-2r^2e^{2q(\gamma_R-\psi_R)}\dot{\theta}\Big] = \frac{\partial\mathscr{L}}{\partial\theta}\,,\,. \tag{24}$$

In addition we have to set

$$\mathscr{L} = \epsilon = \begin{cases} +1 & \text{for timelike geodesics}\,,\\ -1 & \text{for spacelike geodesics}\,,\\ 0 & \text{for null geodesics}\,.\end{cases} \tag{25}$$

We will not consider spacelike geodesics. Taking the derivative of (25) with respect to s, using (24) and assuming $\dot{r}\neq 0$, (23) follows, therefore (23) is a consequence of the other equations.

With (21-25) we have derived all equations that have to be solved in order to determine geodesics. We substitute the expressions for the constants of motion into (25) and write the result[6] as

$$\dot{r}^2 = E^2e^{-2q\gamma_R} - \Big(1-\frac{2m}{r}\Big)\Big[r^2\dot{\theta}^2 + \frac{L^2}{r^2\sin^2\theta}e^{4q\psi_R-2q\gamma_R} + \epsilon\, e^{2q(\psi_R-\gamma_R)}\Big]\,. \tag{26}$$

This equation will be the central equation in the following investigation of geodesics in the equatorial plane.

The chance of finding an analytic solution of the above system of coupled, nonlinear differential equations of second order is very small. We will have to solve them numerically. Additionally, we will use a far-field approximation, based on the assumption that our test particle's trajectory is sufficiently far from the gravitational source. For $\theta=\frac{\pi}{2}$ the Euler-Lagrange equation (24) always holds which leaves us with equation (26).

4.2 Timelike geodesics in the equatorial plane

We set $\theta=\frac{\pi}{2}$ and consider geodesics, for which $r\gg 2m$ applies. The functions in the Erez-Rosen solution can be approximated as

$$e^{2q\psi_R} = 1-\frac{2}{15}q\frac{m^3}{r^3}(3\cos^2\theta-1)+\mathcal{O}\Big(\Big(\frac{m}{r}\Big)^4\Big)\,, \tag{27}$$

$$e^{-2q\gamma_R} = 1+\frac{1}{5}\frac{m^4}{r^4}q\sin^2\theta\big[4-5\sin^2\theta\big]+\mathcal{O}\Big(\Big(\frac{m}{r}\Big)^5\Big)\,. \tag{28}$$

We consider the coordinate r as a function of the angle ϕ. Using

$$\dot{r} = \frac{\mathrm{d}r}{\mathrm{d}\phi}\,\dot{\phi} = L\frac{e^{2q\psi_R}}{r^2}\,r' \quad\text{with}\quad r' := \frac{\mathrm{d}r}{\mathrm{d}\phi}\,,$$

$$u := \frac{2m}{r}\,,\quad \alpha := \frac{2m}{L}\,,$$

[6] There is a misprint in [24] in the expression of the corresponding equation. The factor $\sin^2\theta$ in the last term is incorrect.

and substituting (27) and (28), then (26) becomes

$$(u')^2 = au^3 - u^2 + \epsilon\alpha^2 u - \alpha^2\big(\epsilon - E^2\big)\,, \tag{29}$$

where

$$a = 1 - \alpha^2\tilde{q}\Big(E^2 - \frac{\epsilon}{2}\Big) \quad \text{with} \quad \tilde{q} := \frac{q}{30}\,. \tag{30}$$

We only consider timelike geodesics, hence $\epsilon = 1$. The right side of (29) is a cubic polynomial,

$$F_3(u) = au^3 - u^2 + \alpha^2 u - \alpha^2\big(1 - E^2\big)$$

with roots u_1, u_2 and u_3, and (29) requires $F_3(u) \geq 0$. Thus we obtain "forbidden regions" for u. The positions of the roots determine the geometry of the trajectories. We apply Viète's formulas,

$$u_1u_2u_3 = \frac{\alpha^2}{a}\big(1 - E^2\big)\,, \tag{31}$$

$$u_1 + u_2 + u_3 = \frac{1}{a}\,, \tag{32}$$

$$u_1u_2 + u_1u_3 + u_2u_3 = \frac{\alpha^2}{a}\,. \tag{33}$$

Now we introduce the parameter p and e via

$$u_1 = \frac{2m}{p}(1-e)\,, \quad u_2 = \frac{2m}{p}(1+e)\,, \quad u_3 = \frac{1}{a} - \frac{4m}{p}\,. \tag{34}$$

u_3 is the result of substituting u_1 and u_2 into (32). We now compare

$$(u')^2 = a\Big[u - \frac{2m}{p}(1-e)\Big]\Big[u - \frac{2m}{p}(1+e)\Big]\Big[u - \frac{1}{a} + \frac{4m}{p}\Big]$$

with (29) and obtain a relation between the new parameters p, e and the physical quantities α and E. With the help of (30) we get

$$\alpha^2 = 4\mu\frac{1 - \mu(3+e^2) + 4\tilde{q}\mu^3(1-e^2)^2}{1 - 2\tilde{q}\mu^2(3 + e^2 - 8\mu(1-e^2))} \quad \text{with} \quad \mu := \frac{m}{p}\,, \tag{35}$$

$$\alpha^2(1-E^2) = 4\mu^2(1-e^2)\frac{1 - 4\mu + 2\tilde{q}\mu^2(1-e^2)}{1 - 2\tilde{q}\mu^2(3 + e^2 - 8\mu(1-e^2))}\,. \tag{36}$$

The parameters p, e characterize trajectories that are ellipse-like. Thus we write

$$u(\phi) = 2\mu\big(1 + e\cos\chi(\phi)\big) \tag{37}$$

and substitute this into (29), taking (35) and (36) into account. The differential equation for u becomes

$$\chi' = \sqrt{1 + 2a\mu(e-3)}\sqrt{1 - k^2\cos^2\Big(\frac{\chi}{2}\Big)} = \sqrt{1 - 6a\mu - 2ae\mu\cos\chi(\phi)}\,, \tag{38}$$

$$\text{where} \quad k^2 = \frac{4a\mu e}{1 + 2a\mu(e-3)}\,, \quad a = \frac{1 - 2\tilde{q}\mu + 4\tilde{q}\mu^2(1-e^2)}{1 - 2\tilde{q}\mu^2(3 + e^2 - 8\mu(1-e^2))}\,.$$

This equation can be solved numerically. But it is important to assure that the approximation (29) is valid and the neglected terms are small in comparison to the terms in our approximation. For this reason we want to derive a criterion, that allows us to make a statement about the reliability of the approximation.

Again we approximate the functions in the Erez-Rosen line element, but consider more terms than in (27) and (28), and substitute this into (26). Terms of order > 4 in $u = \frac{2m}{r}$ are neglected. We obtain (29) extended by one more term,

$$(u')^2 = bu^4 + au^3 - u^2 + \alpha^2 u - \alpha^2(1 - E^2) \quad \text{where} \quad b := \frac{5\tilde{q}\alpha^2}{8}(1 - 3E^2)\,. \tag{39}$$

We assume that the Schwarzschild correction au^3 and the Erez-Rosen correction, which is proportional to q, are small in comparison to the Newtonian terms $-u^2 + \alpha^2 u - \alpha^2(1 - E^2)$. Therefore it will be reasonable to demand that

$$|bu^4| \ll |au^3|\,, \tag{40}$$

assuming $a \neq 0$.

In analogy to our previous approach we consider the right side of (39) as a polynomial,

$$F_4(u) := bu^4 + au^3 - u^2 + \alpha^2 u - \alpha^2(1 - E^2)\,,$$

with roots u_1, u_2, u_3 and u_4. Again Viète's formulas hold,

$$\begin{aligned}
-\frac{\alpha^2(1 - E^2)}{b} &= u_1 u_2 u_3 u_4\,, & (41)\\
-\frac{\alpha^2}{b} &= u_1 u_2 u_3 + u_1 u_2 u_4 + u_1 u_3 u_4 + u_2 u_3 u_4\,, &\\
-\frac{1}{b} &= u_1 u_2 + u_1 u_3 + u_2 u_3 + u_1 u_4 + u_2 u_4 + u_3 u_4\,, &\\
-\frac{a}{b} &= u_1 + u_2 + u_3 + u_4\,. & (42)
\end{aligned}$$

We write again

$$u_1 = 2\mu(1 - e)\,, \quad u_2 = 2\mu(1 + e)\,,$$

where $\mu = \frac{m}{p}$. With (41) and (42), we get

$$\begin{aligned}
u_3 &= -\frac{a}{b} - 4\mu - u_4\,,\\
u_4 &= -\Big(\frac{a}{2b} + 2\mu\Big) \pm \sqrt{\Big(\frac{a}{2b} + 2\mu\Big)^2 + \frac{\alpha^2(1 - E^2)}{4\mu^2 b(1 - e^2)}}\,.
\end{aligned}$$

If we compare $b(u-u_1)(u-u_2)(u-u_3)(u-u_4)$ with $F_4(u)$, we get two equations, which relate the parameters α, E with μ, e,

$$\alpha^2 = \frac{4\mu\Big(1 - (3 + e^2)\mu + 4\tilde{q}(e^2 - 1)^2\mu^3 + 30\tilde{q}\mu^5(e^2 - 1)^3\Big)}{1 - 2\tilde{q}\mu^2\Big(3 + e^2 + 4\mu(3 + 7e^2) + 15\mu^2(-3 + 2e^2 + e^4) + 10\tilde{q}\mu^4(e^2 - 1)^3\Big)}\,,$$

$$E^2 = \frac{1 + 2\mu\Big[-2 + (e^2 - 1)\mu\Big(-2 + \tilde{q}\mu(e^2 - 1)\big[1 + 5\mu(1 + \mu(e^2 - 1))\big]\Big)\Big]}{1 - \mu(3 + e^2) + 4\tilde{q}\mu^3(e^2 - 1)^2 + 30\tilde{q}\mu^4(e^2 - 1)^2 + 30\tilde{q}\mu^5(e^2 - 1)^3}\,.$$

Substituting (37) into (39), we obtain a differential equation for $\chi(\phi)$,

$$\chi'(\phi)^2 = -\frac{1}{4e^2\mu^2\sin^2\chi(\phi)}\Big(\alpha^2 - E^2\alpha^2 - 2\alpha^2\mu\big(1+e\cos\chi(\phi)\big) + 4\mu^2\big(1+e\cos\chi(\phi)\big)^2 \\ - 8a\mu^3\big(1+e\cos\chi(\phi)\big)^3 - 16b\mu^4\big(1+e\cos\chi(\phi)\big)^4\Big)\,.$$

In order for (29) to be a valid approximation (40) should hold. Setting the two sides equal, we get a critical value u_{max} (respectively r_{min}). The occurring values of u (respectively r) are not allowed to be greater (smaller) than this upper (lower) limit, since otherwise a numerical solution of (29) cannot be considered a valid approximation and it will differ from the solution of (39) in corresponding plots. The critical value is given by

$$u_{max} = \left|\frac{a}{b}\right| = \left|\frac{8(1-\tilde{q}\alpha^2(E^2-\frac{1}{2}))}{5\tilde{q}\alpha^2(1-3E^2)}\right| \quad \Rightarrow r_{min} = \frac{2m}{u_{max}}\,.$$

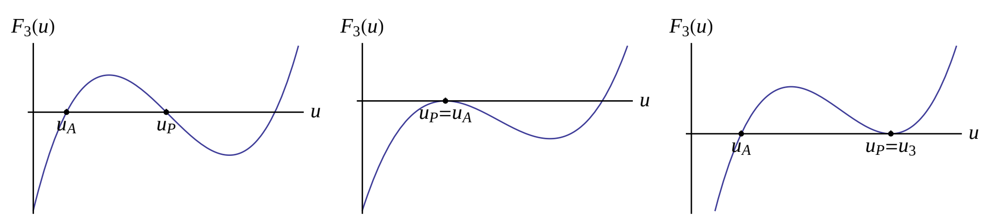
If r_p is the minimal value of r along a geodesic, this value must be greater than r_{min}. Our criterion therefore requires

$$r_p \gg r_{min}\,. \tag{43}$$

As pointed out before, the positions of the roots of $F_3(u)$ determine the geometry of the geodesics. These positions themselves depend on the parameters a and E^2. We have to distinguish between four cases, corresponding to $a \lessgtr 0$ and $E^2 \lessgtr 1$.

4.2.1 Case 1: $a > 0$, $E^2 < 1$: Bound orbits

With Viète's formulas one can show that there is at least one positive root. If the other roots are real, they are positive as well. We assume the existence of three real roots, the qualitative behaviour of $F_3(u)$ is shown in Fig. 3. We



(a) $u_1 \neq u_2$ (b) $u_1 = u_2$ (c) $u_2 = u_3$

Figure 3: Qualitative behaviour of $F_3(u)$ for $a > 0$, $E^2 < 1$.

distinguish between three cases: (a) all roots distinct, (b) $u_1 = u_2 \neq u_3$, (c) $u_1 \neq u_2 = u_3$.

In case (a) two types of orbits are possible: for $u_1 \leq u \leq u_2$, u oscillates between a minimum and a maximum value, it will be an ellipse-like orbit. The maximum value is $u_2 \equiv u_p$, the perihel, the minimal $u_1 \equiv u_a$, the aphel. For $u > u_3$ the test particle will eventually fall to the center. Because of our approximation, we cannot describe such geodesics reliably. Following [3], we call these two characteristic types orbits of the first and second kind respectively. We will only study orbits of the first kind further.

Accordingly the only interesting orbit in (b) will be the circular motion with $u_1 = u_2$.

In case (c) a particle starting in the interval (u_1, u_2) approaches a circular trajectory with the radius $\frac{2m}{u_2}$, for $u > u_2$ we get an orbit of the second kind.

(a): $u_1 \neq u_2$

For $u_1 \neq u_2$, hence $e \neq 0$, the test particle follows a path along which the value of u oscillates, the geodesic is ellipse-like, see Fig. 3(a). u oscillates between u_a and u_p.

The limit u_{max} may not be in this interval, it should be considerably greater than u_p. With

$$u_p = 2\mu(1+e) \quad \Longrightarrow \quad r_p = \frac{p}{(1+e)}$$

our criterion (43) reads

$$\frac{p}{(1+e)} \gg r_{min} \, .$$

If we apply it to the parameters chosen in [24], we notice that only for the smallest value, $\tilde{q} = \frac{1}{30}$, the approximation is valid. For $\tilde{q} = 10$ one even has $r_p < r_{min}$. It is therefore not suprising that the corresponding plots in Fig. 4.b differ from each other. The order of magnitude of the parameters chosen in [24] is partly incompatible with the assumed approximation. This is also true for the following cases presented in [24]. The parameter of the plots in Fig. 5 as well as of all the following plots presented in this thesis fulfil the criterion.

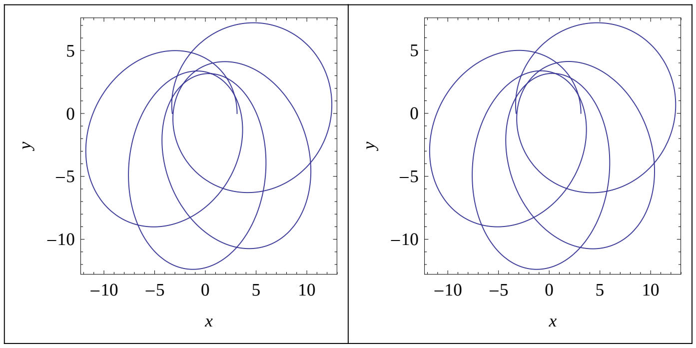
In Fig. 5 we observe that the perihelion shift decreases with increasing Erez-Rosen-parameter. We will show this to be true in 4.2.5.

(b): $u_1 = u_2 =: u_c$

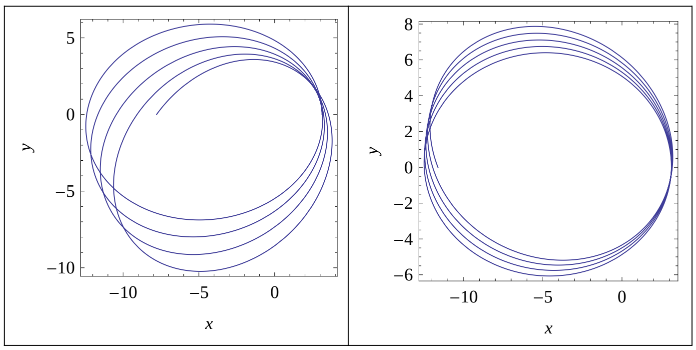
In this case we have $e = 0$ and thus a circular motion with $r_c := \frac{2m}{u_c} = p$, see Fig. 3.b. Substituting $e = 0, \mu = \frac{2m}{r_c}$ into (35), we get

$$r_c^2\Big(r_c^2 - 4\frac{m}{\alpha^2}r_c + 12\frac{m^2}{\alpha^2}\Big) - 6m^2\tilde{q}\Big(r_c^2 - \frac{8}{3}mr_c + \frac{8}{3}\frac{m^2}{\alpha^2}\Big) = 0 \, , \tag{44}$$

which determines the possible radii. The Erez-Rosen deviation of the Schwarzschild solution ($\tilde{q} \neq 0$) gives rise to additional such radii. Our criterion for the reliability of the approximation yields $p \gg r_{min}$.

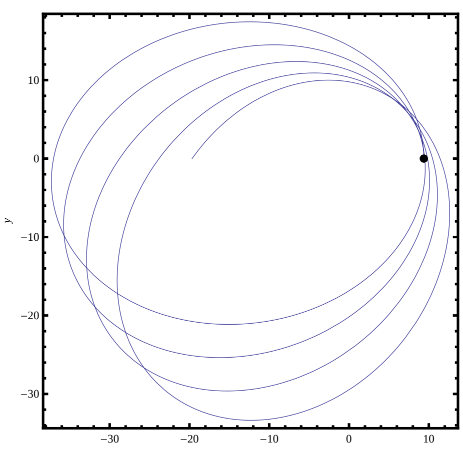


(a) $\tilde{q} = 1/30$

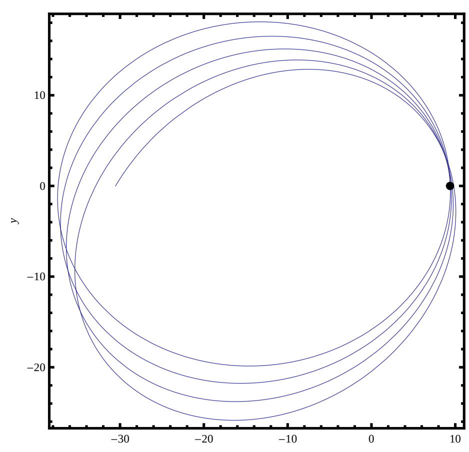


(b) $\tilde{q} = 10$

Figure 4: Comparison of different approximations, geodesics solving (29)(on the left) and (39)(on the right) respectively ($m = \frac{1}{5}, p = 5, e = 0.6$).



(a) $\tilde{q} = 0$



(b) $\tilde{q} = 15$

Figure 5: Ellipse-like geodesics ($m = \frac{1}{5}, e = 0.6, p = 15$).

(c): $u_2 = u_3$

According to (34) this is only possible if

$$\mu = \frac{1}{2a(3+e)}$$

holds. Thus the factor k^2 in (38) becomes unity and the equation reduces to

$$\chi' = -\sqrt{\frac{2e}{3+e}} \sin\left(\frac{\chi}{2}\right).$$

As the particle approaches the perihel ($\chi \to 0 \Rightarrow u \to 2\mu(1+e)$) we have $\chi' \to 0$, the perihel is approached if $\phi \to \infty$, therefore the orbit is asymptotically circular, see Fig. 6.

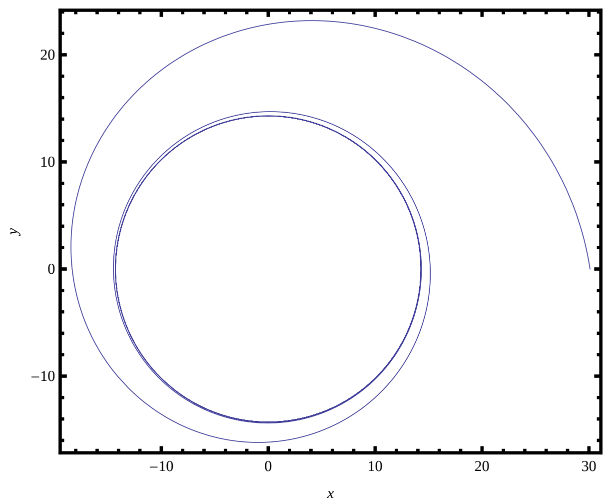
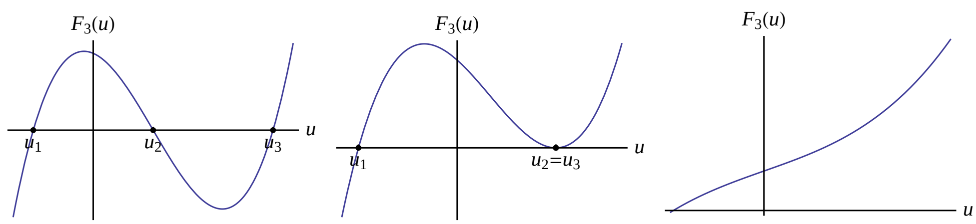


Figure 6: Asymptotically circular geodesics ($m = \frac{1}{5}, \tilde{q} = 5, p = 50, e = 0.4$).

4.2.2 Case 2: $a > 0, E^2 > 1$

According to (31-33), $F_3(u)$ has a negative root. The other two roots are either both positive or a complex conjugate pair, see Fig. 7. In the subcase (c), a test



(a) Two positive roots (b) $u_2 = u_3$ (c) Complex conjugate pair

Figure 7: Qualitative behaviour of $F_3(u)$ for $a > 0, E^2 > 1$.

particle coming from infinity would fall to the center, only orbits of the second kind are possible. Therefore we restrict ourselves to the cases (a) and (b). The motion occurs in the interval $0 < u \leq u_2$. Because of (34) and $u_1 < 0$,

$$e > 1$$

holds.

$0 < a \leq 1$ **or** $a > 1$

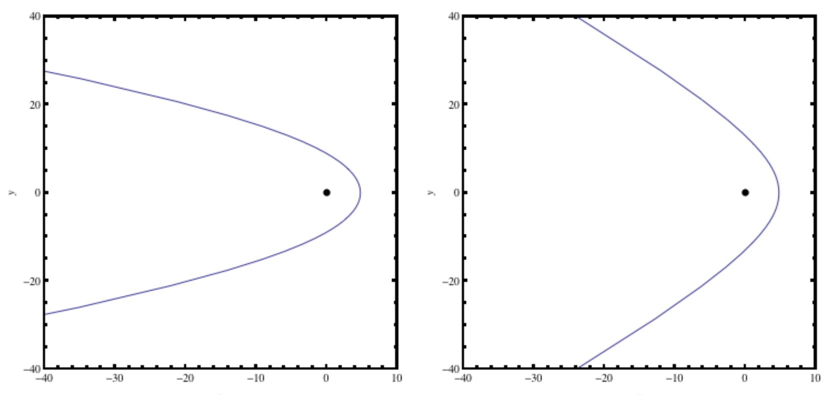
Because of

$$a = 1 - \alpha^2 \tilde{q}\Big(E^2 - \frac{1}{2}\Big) > 1$$

$\tilde{q}$ must be negative if $a > 1$, which corresponds to an oblate mass distribution (cf. 3.1). If $0 < a \leq 1$ the sign of $\tilde{q}$ is indefinite.

(a) $u_2 \neq u_3$

For $u < u_2$ a test particle coming from infinity will approach the perihel and then disappear back to infinity. The path is hyperbolic. Some geodesics, which satisfy our criterion, are shown in Fig. 8.



(a) $\tilde{q} = 2$ (b) $\tilde{q} = 50$

Figure 8: Case 2 (a): Two possible geodesics for different values of $\tilde{q}$ ($m = \frac{1}{5}, e = 1.1, p = 10$).

(b) $u_2 = u_3$

In this case we have

$$a = \frac{1}{2\mu(3+e)} \,.$$

Therefore (38) becomes, as in case 4.2.1 (c),

$$\chi' = -\sqrt{\frac{2e}{3+e}} \sin\left(\frac{\chi}{2}\right) .$$

A test particle, that passes a point with radius greater than $r_2 = \frac{2m}{u_2}$, will move towards infinity or, as shown in Fig. 9, will approach a circular orbit for the same reason as in 4.2.1(c).

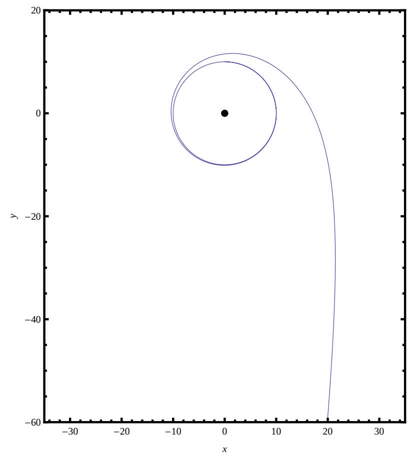
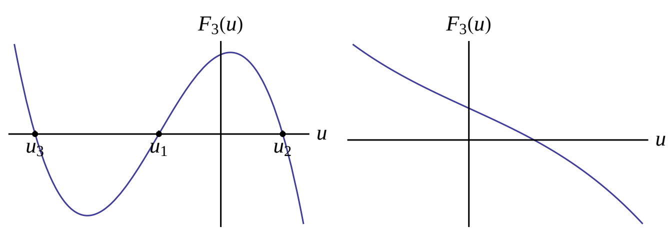


Figure 9: Asymptotic circular motion ($m = \frac{1}{5}, \tilde{q} = 5, p = 25, e = 1.5$).

4.2.3 Case 3: $a < 0, E^2 > 1$

Using Viète's formulas (31-33), $F_3(u)$ has again one positive root. The other roots are either negative or a complex conjugate pair. The behaviour is visualized in Fig. 10. We consider the case (a). For the roots (34)



(a) Two negative roots (b) Complex conjugate pair

Figure 10: Possible behaviour of $F_3(u)$ for $a < 0, E^2 > 1$.

$$u_3 \leq u_1 < 0 < u_2$$

holds, hence

$$e > 1$$

follows. Just as in 4.2.2, the resulting motion is hyperbolic. An example, which fulfils our criterion $r_p \gg r_{min}$, is shown in Fig. 11. Since

$$a = 1 - \alpha^2 \tilde{q}\Big(E^2 - \frac{1}{2}\Big) < 0\,, \quad E^2 > 1\,,$$

$\tilde{q}$ must be positive, according to 3.1 this corresponds to a prolate mass distribution.

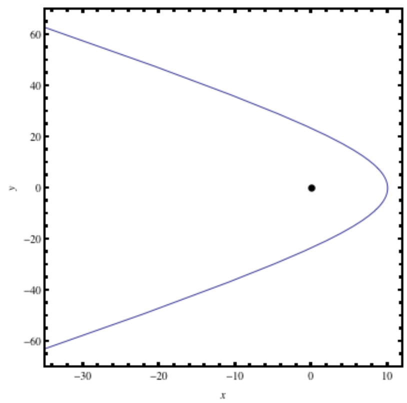
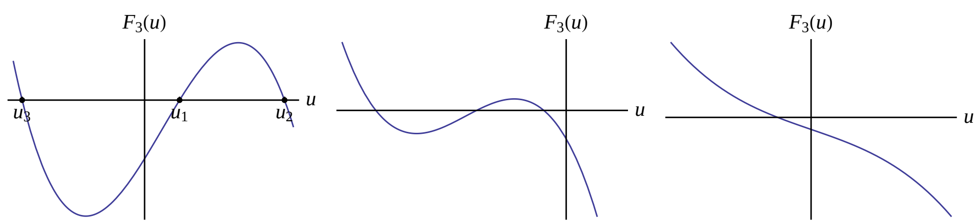


Figure 11: A hyperbolic geodesic ($a < 0, E^2 > 1$, $m = \frac{1}{5}, \tilde{q} = 5, p = 25, e = 1.5$).

4.2.4 Case 4: $a < 0, E^2 < 1$

Using (31-33), we obtain the existence of at least one negative root of $F_3(u)$. The other two are both negative, both positive or a complex conjugate pair, see Fig. 12. The cases (b) and (c) allow no solution, because $F_3(u) < 0$ for all $u > 0$. In the case (a)



(a) Two positive roots (b) Two negative roots (c) Complex conjugate pair

Figure 12: Possible behaviour of $F_3(u)$ for $a < 0, E^2 < 1$.

$$u_3 < 0 < u_1 \leq u_2$$

holds and therefore

$$0 \leq e < 1\,.$$

The geodesics for $u_1 \leq u \leq u_2$ are ellipse-like, analogous to case (a) in 4.2.1. Circular motion occurs for $u_1 = u_2 \Leftrightarrow e = 0$.

4.2.5 Perihelion shift

In order to determine the perihelion shift, we expand (38) in powers of μ to the second order and obtain

$$\mathrm{d}\phi \approx \Big(1 + (3 + e\cos\chi)\mu + \frac{1}{2}(3 + e\cos\chi)(9 - 4\tilde{q} + 3e\cos\chi)\mu^2\Big)\mathrm{d}\chi\,.$$

Integration yields

$$\phi = \frac{1}{4}\Big[\big(4+3\mu\big(4+(18+e^2-8\tilde{q})\mu\big)\big)\chi + e\mu(4+36\mu-8\tilde{q}\mu+3e\mu\cos\chi)\sin\chi\Big]\,,$$

and we obtain the perihelion shift

$$\Delta\phi = \phi(2\pi)-\phi(0)-2\pi = 6\pi\mu + 3\pi\mu^2\big(9+\frac{1}{2}e^2-4\tilde{q}\big)\,.$$

The first term is the perihelion shift in the Schwarzschild spacetime.

As an example we calculate the effect of the Erez-Rosen-Parameter on the perihelion shift of Mercury. With the data for Mercury [31] and the calculated Erez-Rosen parameter (20), we get the following shift per Julian century,

$$-12\pi\mu^2\tilde{q} \approx 0.023''\,.$$

This is a tiny contribution compared to the perihelion shift calculated by Einstein and it is therefore legitimatly neglected.

4.3 Non-planar geodesics

We drop our previous restriction $\theta = \frac{\pi}{2}$ and thus have to consider a more complicated system of equations. We expand $e^{2q\psi}, e^{-2q\psi}, e^{2q\gamma}, e^{2q(\lambda-\psi)}, \psi_r, \psi_\theta, \gamma_r, \gamma_\theta$ and neglect terms of order > 3 in $\frac{2m}{r}$. Substituting this into (23) and (24), we obtain the Euler-Lagrange-equations,

$$\ddot{u} = f_S + q\, f_{ER}\,, \quad \ddot{\theta} = g_S + q\, g_{ER}\,, \tag{45}$$

where
$$f_S = \frac{1}{8m^2}\left[\frac{L^2}{2m^2\sin^2\theta}(u-1)u - \frac{E^2}{u-1}\right] + \left[\frac{5}{2}-\frac{2}{u}\right]\frac{\dot{u}^2}{u-1} + (u-1)u\,\dot{\theta}^2\,,$$
$$f_{ER} = \frac{u^3}{40}\left[-\frac{L^2}{m^4\sin^2\theta}\,\frac{5(1+3\cos 2\theta)(u-1)u^5}{120+q(1+3\cos 2\theta)u^3} + 2\sin 2\theta\,\dot{u}\,\dot{\theta}\right],$$
$$g_S = \frac{L^2\cos\theta}{16m^4\sin^3\theta}u^4 + \frac{2}{u}\,\dot{u}\,\dot{\theta}\,,$$
$$g_{ER} = \frac{1}{320m^4}u\Bigg(-\frac{40L^2\cos\theta\, u^6}{\sin^3\theta(30+qu^3)} - \frac{180L^2q^2\sin 2\theta\, u^{12}}{\big(30+qu^3\big)^2\big(120+q(1+3\cos 2\theta)u^3\big)} + \frac{L^2\cos\theta\, u^6\big(900+q^2u^6\big)}{\sin\theta\big(30+qu^3\big)^2} + \frac{2m^2\sin 2\theta\big(E^2u^4+4m^2\dot{u}^2+4m^2(u-1)u^2\dot{\theta}^2\big)}{u-1}\Bigg)\,.$$

4.3.1 Non-existence of non-planar spherical geodesics

Motivated by the existence of non-planar spherical photon orbits in the Kerr metric [28], we look for similar geodesics in the Erez-Rosen spacetime. For spherical geodesics the radial coordinate r is constant, therefore

$$r := r_c = \text{Const.} \Rightarrow \dot{r} = \ddot{r} = 0\,.$$

In this case the geodesic equation (24) automatically follows from (25). The equation (23) becomes

$$\frac{\partial \mathscr{L}}{\partial r} = 0\,. \tag{46}$$

With (21) and (22), $\mathscr{L} = \epsilon$ yields

$$\frac{E^2}{1-\frac{2m}{r_c}} e^{2q\psi_R} - \frac{L^2}{r_c^2 \sin^2\theta} e^{-2q\psi_R} - e^{2q(\gamma_R - \psi_R)} r_c^2 \dot{\theta}^2 = \epsilon$$

and (46) gives us

$$\Big[1 + q r_c \Big(\frac{\partial \gamma_R}{\partial r}\Big|_{r=r_c} - \frac{\partial \psi_R}{\partial r}\Big|_{r=r_c} \Big)\Big] e^{2q(\gamma_R - \psi_R)} r_c \dot{\theta}^2 + \Big[\frac{1}{r_c} - q \frac{\partial \psi_R}{\partial r}\Big|_{r=r_c}\Big] \frac{L^2}{r_c^2 \sin^2\theta} e^{2q\psi_R}$$
$$-\Big[m + q r_c^2 \frac{\partial \psi_R}{\partial r}\Big|_{r=r_c} \Big(1 - \frac{2m}{r_c}\Big)\Big] \frac{E^2}{r_c^2 (1-2m/r_c)^2} e^{-2q\psi_R} = 0\,,$$

a second equation for $\dot{\theta}^2$. We eliminate $\dot{\theta}^2$ and get a condition for possible radii r_c,

$$\frac{L^2}{r_c^2 \sin^2\theta}\Big[\frac{1}{r_c} + e^{4q\psi_R}\Big(-\frac{1}{r_c} + q \frac{\partial \psi_R}{\partial r}\Big|_{r=r_c}\Big)\Big] - \frac{E^2}{r_c(1-2m/r_c)}\Big[e^{-4q\psi_R} - \frac{m}{r_c(1-2m/r_c)}\Big]$$
$$+ q\Big[\Big(\frac{\partial \gamma_R}{\partial r}\Big|_{r=r_c} - \frac{\partial \psi_R}{\partial r}\Big|_{r=r_c}\Big)\Big(\frac{L^2}{r_c^2 \sin^2\theta} + \epsilon e^{2q\psi_R} - E^2 e^{4q\psi_R}\Big) + \frac{E^2}{1-2m/r_c} \frac{\partial \psi_R}{\partial r}\Big|_{r=r_c}\Big]$$
$$+ \frac{\epsilon}{r_c} e^{2q\psi_R} = 0\,. \tag{47}$$

For $q \neq 0$ this equation depends on θ. Consequently the only geodesics on a sphere are circular. The only physically reasonable value for the constant coordinate θ is $\frac{\pi}{2}$. But that is just a geodesic movement in the equatorial plane which was discussed in 4.2.

4.3.2 Perturbation of circular equatorial orbits

In 4.2.1 we saw that equatorial circular geodesics are possible. Their radii were determined by (44). For a given mass m, quadrupole moment q and angular momentum L, we obtain possible radii of circular motion. The only parameter left aside at this point is the energy E. We take the equation (47), set $\epsilon = 1$ for timelike geodesics, $\theta = \frac{\pi}{2}$ for equatorial orbits and substitute the series expansion of $e^{\pm 4q\psi_R}$, $e^{2q\psi_R}$, and the derivatives of the functions γ_R and ψ_R. We consider terms up to the third power in $\frac{2m}{r}$. We obtain

$$\frac{2m^3 q}{15 r_c^5} + \frac{1}{r_c^2} + \frac{3E^2}{4m r_c} + \frac{E^2(8m - 3r_c)}{4m(r_c - 2m)^2} = 0\,.$$

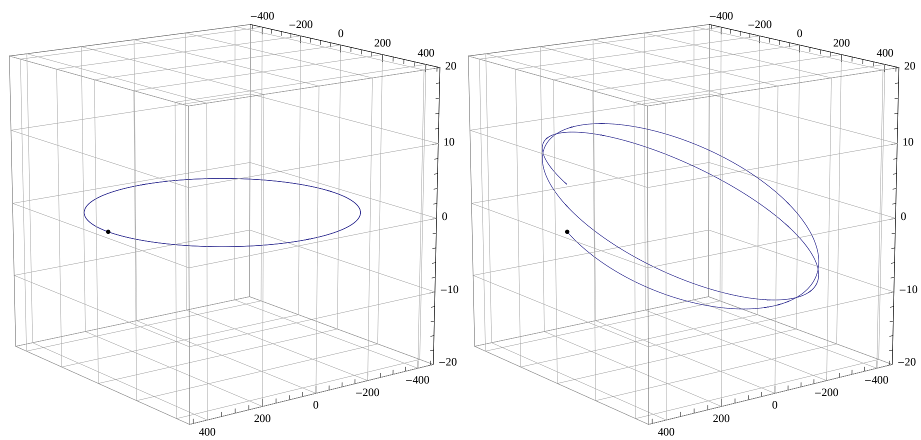
This gives us an expression for the energy of a circular geodesic,

$$E^2 = \frac{(r_c - 2m)^2 (2m^3 q + 15 r_c^3)}{15 (r_c - 3m) r_c^4}\,.$$

With this choice of the physical parameters we can solve (45) numerically. Instead of s, we use ϕ as a parameter. For the initial values

$$u(0) = \frac{2m}{r_c}\,, \quad u'(0) := \left.\frac{\mathrm{d}u(\phi)}{\mathrm{d}\phi}\right|_{\phi=0} = 0\,,$$
$$\theta(0) = \frac{\pi}{2}\,, \quad \theta'(0) := \left.\frac{\mathrm{d}\theta(\phi)}{\mathrm{d}\phi}\right|_{\phi=0} = 0\,,$$

we obtain circular geodesics as plotted in Fig. 13(a). Now we can pertub this



(a) $v_\theta = 0$ (b) $v_\theta = 0.02$

Figure 13: A circular equatorial geodesic and a non-planar perturbation ($m = 1, q = 50000, L = 20, E \approx 1, r_c \approx 420$).

motion by modifying the inital value for $\theta'(0)$ to

$$\theta'(0) = v_\theta \neq 0\,,$$

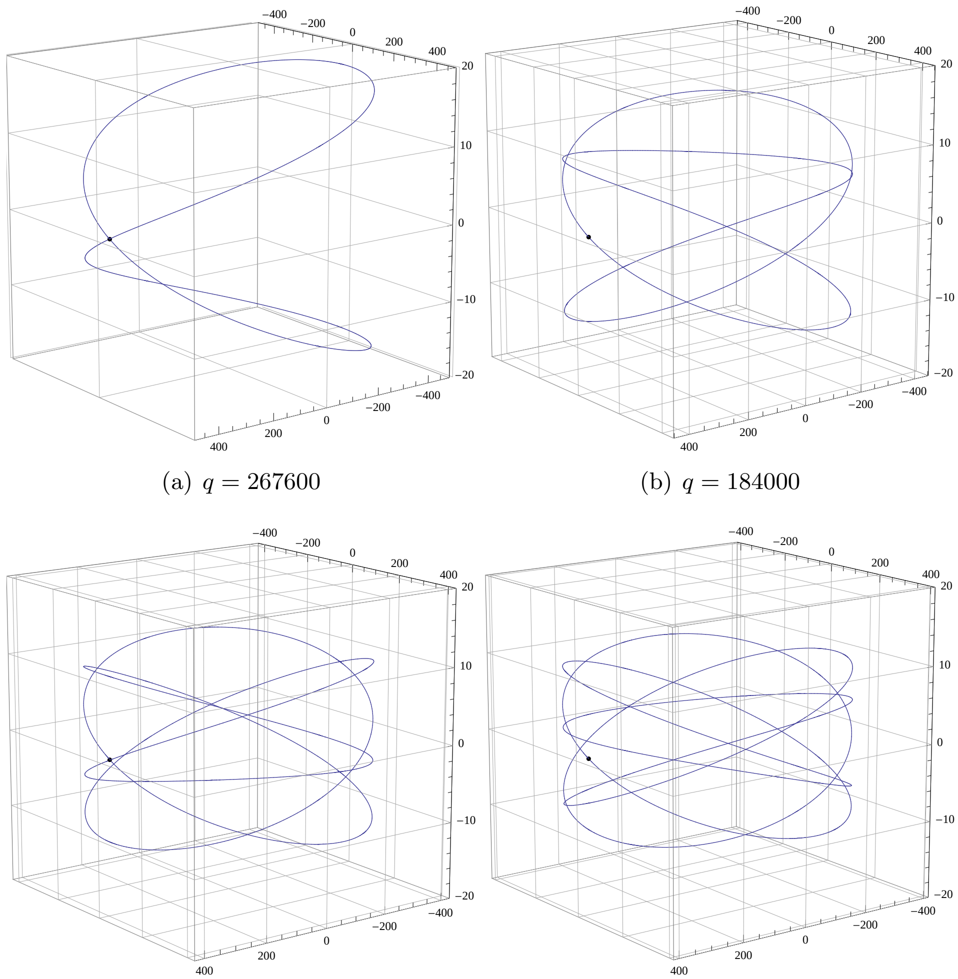
with the result shown in Fig. 13(b). Note that the deviation from the equatorial plane of the geodesic in Fig. 13(b) is a few degrees only.

In order to observe an effect caused by the quadrupole moment, we have to look at geodesics close to the gravitational source. That is why we do not use the equations (45) but improve it with terms up of the fourth order in $\frac{2m}{r}$. Just as in 4.2 we always compare a plot to the resulting plot of a higher-order approximation. If they differ, the parameters or initial values are not reliable.

The most important difference to the geodesics in the Schwarzschild spacetime is the fact that the geodesics are necessarily non-planar (except for equatorial and polar orbits).

Closed geodesics

By fixing the parameters m and L, as well as the velocity v_θ, and precisely adjusting the value of q, we can obtain closed geodesics. After n revolutions the test particle gets back to the starting point with the same velocity. If we want n to be small, we have to choose a high value for q, thus increasing the effect of the quadrupole moment on the geodesic. Closed geodesics with $n = 2, 3, 4, 5$ are visualized in Fig. 14.



(a) $q = 267600$ (b) $q = 184000$

(c) $q = 138100$ (d) $q = 109900$

Figure 14: Closed geodesics for $m = 1, L = 18, v_\theta = 0.02$.

For this adjustment of parameters ($m = 1, L = 18, v_\theta = 0.02$) the value of q that is approximately required for the existence of a closed geodesic after n revolutions is presented in Fig. 15. Analytical explorations of such orbits are desired. Further thoughts on this follow in section 5.

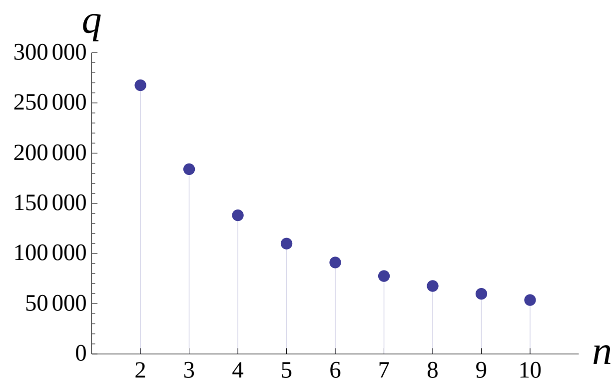


Figure 15: Correlation of q and n for closed geodesics ($m = 1, L = 18, v_\theta = 0.02$).

5 Conclusions and discussion

We were able to reproduce the results about equatorial orbits in the Erez-Rosen spacetime obtained by Quevedo and Parkes [24] and made some further observations regarding the far-field approximation. As expected the equatorial orbit's deviation from the Schwarzschild geodesics is typically negligible. For example the perihelion shift of Mercury's orbit caused by the sun's flatness is tiny as shown in 4.2.5. Nonetheless with a calculated value of $\Delta\phi \approx 0.02''$ per Julian century it is not beyond the possibility of experimental observation. With the measured result of $(43.0115 \pm 0.0085)''$ per century [18] it should be possible to extract the quadrupole effect. Considering the magnitude of its contribution a precise knowledge about some other effects would be necessary. By precise observations of orbits of planets or space craft one could be able to measure the sun's oblateness and quadrupole moment indirectly.

In this work we started to investigate non-planar geodesics. Motivated by the surprising results in [28], we looked for non-planar spherical geodesics in the Erez-Rosen spacetime. We found that this kind of motion is not possible. More generally there are no geodesics on spheroidal hypersurfaces.

Further investigations were concerned with perturbations of circular equatorial orbits. This results in a non-planar trajectory oscillating about the equatorial plane. By tuning the parameters it was possible to obtain approximately closed geodesics. Here we had to choose large values for q in order to visualize the effect.

These kinds of orbits around an oblate spheroidal mass distribution have been known for a long time in Newtonian gravity [15, 27]. In order to tell which fraction of our results can be attributed to relativistic , and thus beyond Newtonian effects, further investigations are necessary. An analytical access to this kind of geodesics is complicated and requires higher mathematical methods.

By restricting our investigations to static spacetimes, we neglected a possible rotation of the gravitational source. A generalization of the Erez-Rosen solution was presented by Quevedo and Mashhoon [23] and Castejon-Amenedo and Manko [2], which possesses not only an arbitrary multipole structure but also rotation. It contains the Erez-Rosen and the Kerr solution as special cases.

Therefore and since most observed astrophysical objects indeed rotate we should consider our investigations as an approximation for slow rotation. In the case of the sun this assumption may be justifiable. But in order to get more realistic results, e.g. for planetary orbits around a fast rotating star, one could adapt our approach to the Quevedo-Mashhoon solution. A similar but not identical solution of the Einstein equations was found by Manko and Novikov [17], with an arbitrary set of multipoles and rotation. There have been investigations of geodesics in this spacetime [5].

Another problem concerns the Newtonian limit in 3.1, where we interpreted the Erez-Rosen solution as a description of the vacuum around a flattened star. We even derived a relation between the star's geometrical proportions and the quadrupole moment q. For this conclusion to be fully consistent, one needs an interior solution that can be smoothly matched to the Erez-Rosen solution. Interior solutions with a not to unphysical stress-energy tensor for the Weyl class were presented by Hernandez [10], with an emphasis on the Erez-Rosen solution. Furthermore, Quevedo presented an approximate interior solution matching the Quevedo-Mashhoon solution in the limiting case of slightly deformed masses and slow rotation [22].

In the Newtonian limit a matching interior solution should reduce to the Newtonian potential of a homogenous spheroid considered in 3.1.2. Only then the Erez-Rosen solution could be fully acceptable as the vacuum around a flattened star.

Furthermore, we have seen, that the Erez-Rosen solution has a high order multipole structure in the Geroch-Hansen sense. As described in 3.2, there is a solution with a GH-monopole and GH-quadrupole moment only. An interior metric for one of the approximative $(M - Q)$ solutions has been obtained via Hernandez' results in [12]. A study of the geodesics in this spacetime has been presented in [11]. This solution is an alternative to describe the spacetime around a spheroidal mass distribution.

Some computations in this work have been carried out with the help of Mathematica.

A Weyl metric with time-dependence

We calculate the curvature and the field equation of the Weyl spacetime, where we allow an additional time-dependence of the functions ψ and γ. The line element is given by

$$\mathrm{d}s^2 = e^{2\psi}\mathrm{d}t^2 - e^{2(\gamma-\psi)}\left(\mathrm{d}\rho^2 + \mathrm{d}z^2\right) - \rho^2 e^{-2\psi}\mathrm{d}\phi^2 \,.$$

ψ and γ are functions of ρ, z and t. Weyl showed that, without the t-dependence, a static axially symmetric metric can be written in this form [29].

Orthonormal basis

We are looking for a coframe field θ^i such that

$$g = g_{\mu\nu}\mathrm{d}x^\mu \otimes \mathrm{d}x^\nu = \eta_{ij}\theta^i \otimes \theta^j \,,$$

where η_{ij} are the components of the Minkowski metric (as in an inertial frame). Since the metric is diagonal, an obvious choice is

$$\theta^0 = e^{\psi}\mathrm{d}t \,, \quad \theta^1 = e^{\gamma-\psi}\mathrm{d}\rho \,, \quad \theta^2 = e^{\gamma-\psi}\mathrm{d}z \,, \quad \theta^3 = \rho e^{-\psi}\mathrm{d}\phi \,.$$

For the differentials of these 1-forms we obtain

$$\begin{aligned}
\mathrm{d}\theta^0 &= -e^{-(\gamma-\psi)}\left(\frac{\partial\psi}{\partial\rho}\,\theta^0\wedge\theta^1 + \frac{\partial\psi}{\partial z}\,\theta^0\wedge\theta^2\right) \,, \\
\mathrm{d}\theta^1 &= \frac{\partial(\gamma-\psi)}{\partial t}e^{-\psi}\,\theta^0\wedge\theta^1 - \frac{\partial(\gamma-\psi)}{\partial z}e^{-(\gamma-\psi)}\,\theta^1\wedge\theta^2 \,, \\
\mathrm{d}\theta^2 &= \frac{\partial(\gamma-\psi)}{\partial t}e^{-\psi}\,\theta^0\wedge\theta^2 + \frac{\partial(\gamma-\psi)}{\partial \rho}e^{-(\gamma-\psi)}\,\theta^1\wedge\theta^2 \,, \\
\mathrm{d}\theta^3 &= -\frac{\partial\psi}{\partial t}e^{-\psi}\,\theta^0\wedge\theta^3 + \left(\frac{1}{\rho} - \frac{\partial\psi}{\partial\rho}\right)e^{-(\gamma-\psi)}\,\theta^1\wedge\theta^3 - \frac{\partial\psi}{\partial z}e^{-(\gamma-\psi)}\,\theta^2\wedge\theta^3 \,.
\end{aligned}$$

Connection 1-forms

The connection is the Levi-Civita connection, which is uniquely determined by vanishing torsion and metric compatibility.

Vanishing torsion

This is expressed by

$$\mathrm{d}\theta^i + \omega^i{}_j\wedge\theta^j = 0\,.$$

A solution is given by

$$
\begin{aligned}
\omega^0{}_0 &= f_{00}\theta^0\,, \quad \omega^0{}_1 = e^{-(\gamma-\psi)}\frac{\partial\psi}{\partial\rho}\theta^0 + f_{01}\theta^1\,,\\
\omega^0{}_2 &= e^{-(\gamma-\psi)}\frac{\partial\psi}{\partial z}\theta^0 + f_{02}\theta^2\,, \quad \omega^0{}_3 = f_{03}\theta^3\,,\\
\omega^1{}_0 &= f_{10}\theta^0 + \frac{\partial(\gamma-\psi)}{\partial t}e^{-\psi}\theta^1\,, \quad \omega^1{}_1 = f_{11}\theta^1\,,\\
\omega^1{}_2 &= \frac{\partial(\gamma-\psi)}{\partial z}e^{-(\gamma-\psi)}\theta^1 + f_{12}\theta^2\,, \quad \omega^1{}_3 = f_{13}\theta^3\,,\\
\omega^2{}_0 &= f_{20}\theta^0 + \frac{\partial(\gamma-\psi)}{\partial t}e^{-\psi}\theta^2\,, \quad \omega^2{}_1 = f_{21}\theta^1 + \frac{\partial(\gamma-\psi)}{\partial\rho}e^{-(\gamma-\psi)}\theta^2\,,\\
\omega^2{}_2 &= f_{22}\theta^2\,, \quad \omega^2{}_3 = f_{23}\theta^3\,,\\
\omega^3{}_0 &= -\frac{\partial\psi}{\partial t}e^{-\psi}\theta^3\,, \quad \omega^3{}_1 = \left(\frac{1}{\rho} - \frac{\partial\psi}{\partial\rho}\right)e^{-(\gamma-\psi)}\theta^3\,,\\
\omega^3{}_2 &= -\frac{\partial\psi}{\partial z}e^{-(\gamma-\psi)}\theta^3\,, \quad \omega^3{}_3 = f_{33}\theta^3\,.
\end{aligned}
$$

The f_{ij} are unknown functions that will be fixed by the metric compatibility condition.

Metric compatibility

Relative to the orthonormal basis, the metric compatibility is equivalent to the antisymmetry

$$\omega_{ij} = -\omega_{ji}\,.$$

From this we can identify the functions f_{ij} and obtain the Levi-Civita connection 1-forms,

$$
\begin{aligned}
&\omega^0{}_0 = 0\,, \quad \omega^0{}_1 = e^{-(\gamma-\psi)}\frac{\partial\psi}{\partial\rho}\theta^0 + \frac{\partial(\gamma-\psi)}{\partial t}e^{-\psi}\theta^1\,,\\
&\omega^0{}_2 = e^{-(\gamma-\psi)}\frac{\partial\psi}{\partial z}\theta^0 + \frac{\partial(\gamma-\psi)}{\partial t}e^{-\psi}\theta^2\,, \quad \omega^0{}_3 = -\frac{\partial\psi}{\partial t}e^{-\psi}\theta^3\,,\\
&\omega^1{}_0 = \frac{\partial\psi}{\partial\rho}e^{-(\gamma-\psi)}\theta^0 + \frac{\partial(\gamma-\psi)}{\partial t}e^{-\psi}\theta^1\,, \quad \omega^1{}_1 = 0\,,\\
&\omega^1{}_2 = \frac{\partial(\gamma-\psi)}{\partial z}e^{-(\gamma-\psi)}\theta^1 - \frac{\partial(\gamma-\psi)}{\partial\rho}e^{-(\gamma-\psi)}\theta^2\,, \quad \omega^1{}_3 = \left(\frac{\partial\psi}{\partial\rho} - \frac{1}{\rho}\right)e^{-(\gamma-\psi)}\theta^3\,,\\
&\omega^2{}_0 = \frac{\partial\psi}{\partial z}e^{-(\gamma-\psi)}\theta^0 + \frac{\partial(\gamma-\psi)}{\partial t}e^{-\psi}\theta^2\,,\\
&\omega^2{}_1 = -\frac{\partial(\gamma-\psi)}{\partial z}e^{-(\gamma-\psi)}\theta^1 + \frac{\partial(\gamma-\psi)}{\partial\rho}e^{-(\gamma-\psi)}\theta^2\,,\\
&\omega^2{}_2 = 0\,, \quad \omega^2{}_3 = \frac{\partial\psi}{\partial z}e^{-(\gamma-\psi)}\theta^3\,,\\
&\omega^3{}_0 = -\frac{\partial\psi}{\partial t}e^{-\psi}\theta^3\,, \quad \omega^3{}_1 = \left(\frac{1}{\rho} - \frac{\partial\psi}{\partial\rho}\right)e^{-(\gamma-\psi)}\theta^3\,,\\
&\omega^3{}_2 = -\frac{\partial\psi}{\partial z}e^{-(\gamma-\psi)}\theta^3\,, \quad \omega^3{}_3 = 0\,.
\end{aligned}
$$

Curvature 2-forms and Riemann tensor

With the preceding results and the definition of the curvature

$$\Omega^i{}_j = \mathrm{d}\omega^i{}_j + \omega^i{}_k \wedge \omega^k{}_j = \frac{1}{2} R^i{}_{jkl}\, \theta^k \wedge \theta^l \,,$$

we find

$$\begin{aligned}
\Omega^0{}_1 = & \Bigg[\left(\frac{\partial(\gamma-\psi)}{\partial\rho}\frac{\partial\psi}{\partial\rho} - \frac{\partial(\gamma-\psi)}{\partial z}\frac{\partial\psi}{\partial z} - \frac{\partial^2\psi}{\partial\rho^2} - \left(\frac{\partial\psi}{\partial\rho}\right)^2 \right) e^{-2(\gamma-\psi)} \\
& + \left(\frac{\partial^2(\gamma-\psi)}{\partial t^2} - \frac{\partial(\gamma-\psi)}{\partial t}\frac{\partial\psi}{\partial t} + \left(\frac{\partial(\gamma-\psi)}{\partial t}\right)^2 \right) e^{-2\psi} \Bigg] \theta^0 \wedge \theta^1 \\
& + \left[\left(\frac{\partial(\gamma-\psi)}{\partial z}\frac{\partial\psi}{\partial\rho} + \frac{\partial(\gamma-\psi)}{\partial\rho}\frac{\partial\psi}{\partial z} - \frac{\partial^2\psi}{\partial\rho\partial z} - \frac{\partial\psi}{\partial\rho}\frac{\partial\psi}{\partial z} \right) e^{-2(\gamma-\psi)} \right] \theta^0 \wedge \theta^2 \\
& + \left[\left(\frac{\partial(\gamma-\psi)}{\partial t}\frac{\partial\psi}{\partial z} - \frac{\partial^2(\gamma-\psi)}{\partial t\partial z} \right) e^{-\gamma} \right] \theta^1 \wedge \theta^2 \,, \\
\Omega^0{}_2 = & \left[\left(\frac{\partial(\gamma-\psi)}{\partial\rho}\frac{\partial\psi}{\partial z} + \frac{\partial(\gamma-\psi)}{\partial z}\frac{\partial\psi}{\partial\rho} - \frac{\partial^2\psi}{\partial\rho\partial z} - \frac{\partial\psi}{\partial\rho}\frac{\partial\psi}{\partial z} \right) e^{-2(\gamma-\psi)} \right] \theta^0 \wedge \theta^1 \\
& + \Bigg[\left(\frac{\partial(\gamma-\psi)}{\partial z}\frac{\partial\psi}{\partial z} - \frac{\partial(\gamma-\psi)}{\partial\rho}\frac{\partial\psi}{\partial\rho} - \frac{\partial^2\psi}{\partial z^2} - \left(\frac{\partial\psi}{\partial z}\right)^2 \right) e^{-2(\gamma-\psi)} \\
& + \left(\frac{\partial^2(\gamma-\psi)}{\partial t^2} + \left(\frac{\partial(\gamma-\psi)}{\partial t}\right)^2 - \frac{\partial(\gamma-\psi)}{\partial t}\frac{\partial\psi}{\partial t} \right) e^{-2\psi} \Bigg] \theta^0 \wedge \theta^2 \\
& + \left[\left(\frac{\partial^2(\gamma-\psi)}{\partial t\partial\rho} - \frac{\partial(\gamma-\psi)}{\partial t}\frac{\partial\psi}{\partial\rho} \right) e^{-\gamma} \right] \theta^1 \wedge \theta^2 \,, \\
\Omega^0{}_3 = & \left[\left(\left(\frac{\partial\psi}{\partial z}\right)^2 - \left(\frac{1}{\rho} - \frac{\partial\psi}{\partial\rho}\right)\frac{\partial\psi}{\partial\rho} \right) e^{-2(\gamma-\psi)} + \left(2\left(\frac{\partial\psi}{\partial t}\right)^2 - \frac{\partial^2\psi}{\partial t^2} \right) e^{-2\psi} \right] \theta^0 \wedge \theta^3 \\
& + \left[\left(\frac{\partial\psi}{\partial t}\frac{\partial\psi}{\partial\rho} - \frac{\partial^2\psi}{\partial t\partial\rho} - \left(\frac{1}{\rho} - \frac{\partial\psi}{\partial\rho}\right)\frac{\partial\gamma}{\partial t} \right) e^{-\gamma} \right] \theta^1 \wedge \theta^3 \\
& + \left[\left(\frac{\partial\psi}{\partial t}\frac{\partial\psi}{\partial z} - \frac{\partial^2\psi}{\partial t\partial z} + \frac{\partial\gamma}{\partial t}\frac{\partial\psi}{\partial z} \right) e^{-\gamma} \right] \theta^2 \wedge \theta^3 \,, \\
\Omega^1{}_2 = & \left[\left(\frac{\partial^2(\gamma-\psi)}{\partial t\partial z} - \frac{\partial(\gamma-\psi)}{\partial t}\frac{\partial\psi}{\partial z} \right) e^{-\gamma} \right] \theta^0 \wedge \theta^1 \\
& + \left[\left(\frac{\partial(\gamma-\psi)}{\partial t}\frac{\partial\psi}{\partial\rho} - \frac{\partial^2(\gamma-\psi)}{\partial t\partial\rho} \right) e^{-\gamma} \right] \theta^0 \wedge \theta^2 \\
& + \left[\left(-\frac{\partial^2(\gamma-\psi)}{\partial\rho^2} - \frac{\partial^2(\gamma-\psi)}{\partial z^2} \right) e^{-2(\gamma-\psi)} + \left(\frac{\partial(\gamma-\psi)}{\partial t}\right)^2 e^{-2\psi} \right] \theta^1 \wedge \theta^2 \,,
\end{aligned}$$

$$
\begin{aligned}
\Omega^1{}_3 = &\left[\left(\frac{\partial^2\psi}{\partial t\partial\rho} + \left(\frac{1}{\rho} - \frac{\partial\psi}{\partial\rho}\right)\frac{\partial\gamma}{\partial t} - \frac{\partial\psi}{\partial t}\frac{\partial\psi}{\partial\rho}\right)e^{-\gamma}\right]\theta^0\wedge\theta^3 \\
&+ \left[\left(\frac{\partial^2\psi}{\partial\rho^2} - \frac{\partial\gamma}{\partial\rho}\frac{\partial\psi}{\partial\rho} + \frac{1}{\rho}\frac{\partial\gamma}{\partial\rho} + \frac{1}{\rho}\frac{\partial\psi}{\partial\rho} + \frac{\partial(\gamma-\psi)}{\partial z}\frac{\partial\psi}{\partial z}\right)e^{-2(\gamma-\psi)}\right]\theta^1\wedge\theta^3 \\
&+ \left[\left(\frac{\partial^2\psi}{\partial\rho\partial z} + \left(\frac{1}{\rho} - \frac{\partial\psi}{\partial\rho}\right)\frac{\partial\gamma}{\partial z} - \frac{\partial(\gamma-\psi)}{\partial\rho}\frac{\partial\psi}{\partial z}\right)e^{-2(\gamma-\psi)}\right]\theta^2\wedge\theta^3\,, \\
\Omega^2{}_3 = &\left[\left(\frac{\partial^2\psi}{\partial t\partial z} - \frac{\partial\gamma}{\partial t}\frac{\partial\psi}{\partial z} - \frac{\partial\psi}{\partial t}\frac{\partial\psi}{\partial z}\right)e^{-\gamma}\right]\theta^0\wedge\theta^3 \\
&+ \left[\left(\frac{\partial^2\psi}{\partial\rho\partial z} - \frac{\partial(\gamma-\psi)}{\partial\rho}\frac{\partial\psi}{\partial z} + \left(\frac{1}{\rho} - \frac{\partial\psi}{\partial\rho}\right)\frac{\partial\gamma}{\partial z}\right)e^{-2(\gamma-\psi)}\right]\theta^1\wedge\theta^3 \\
&+ \left[\left(\frac{\partial^2\psi}{\partial z^2} - \frac{\partial\gamma}{\partial z}\frac{\partial\psi}{\partial z} - \left(\frac{1}{\rho} - \frac{\partial\psi}{\partial\rho}\right)\frac{\partial(\gamma-\psi)}{\partial\rho}\right)e^{-2(\gamma-\psi)} + \left(-\frac{\partial(\gamma-\psi)}{\partial t}\frac{\partial\psi}{\partial t}\right)e^{-2\psi}\right]\theta^2\wedge\theta^3\,.
\end{aligned}
$$

Now we read off the non-vanishing components of the Riemann tensor.

$$
\begin{aligned}
R^0{}_{101} = &\left(\frac{\partial(\gamma-\psi)}{\partial\rho}\frac{\partial\psi}{\partial\rho} - \frac{\partial(\gamma-\psi)}{\partial z}\frac{\partial\psi}{\partial z} - \frac{\partial^2\psi}{\partial\rho^2} - \left(\frac{\partial\psi}{\partial\rho}\right)^2\right)e^{-2(\gamma-\psi)} \\
&+ \left(\frac{\partial^2(\gamma-\psi)}{\partial t^2} - \frac{\partial(\gamma-\psi)}{\partial t}\frac{\partial\psi}{\partial t} + \left(\frac{\partial(\gamma-\psi)}{\partial t}\right)^2\right)e^{-2\psi}\,, \\
R^0{}_{102} = &\left(\frac{\partial(\gamma-\psi)}{\partial z}\frac{\partial\psi}{\partial\rho} + \frac{\partial(\gamma-\psi)}{\partial\rho}\frac{\partial\psi}{\partial z} - \frac{\partial^2\psi}{\partial\rho\partial z} - \frac{\partial\psi}{\partial\rho}\frac{\partial\psi}{\partial z}\right)e^{-2(\gamma-\psi)}\,, \\
R^0{}_{112} = &\left(\frac{\partial(\gamma-\psi)}{\partial t}\frac{\partial\psi}{\partial z} - \frac{\partial^2(\gamma-\psi)}{\partial t\partial z}\right)e^{-\gamma}\,, \\
R^0{}_{201} = &\left(\frac{\partial(\gamma-\psi)}{\partial\rho}\frac{\partial\psi}{\partial z} + \frac{\partial(\gamma-\psi)}{\partial z}\frac{\partial\psi}{\partial\rho} - \frac{\partial^2\psi}{\partial\rho\partial z} - \frac{\partial\psi}{\partial\rho}\frac{\partial\psi}{\partial z}\right)e^{-2(\gamma-\psi)}\,, \\
R^0{}_{202} = &\left(\frac{\partial(\gamma-\psi)}{\partial z}\frac{\partial\psi}{\partial z} - \frac{\partial(\gamma-\psi)}{\partial\rho}\frac{\partial\psi}{\partial\rho} - \frac{\partial^2\psi}{\partial z^2} - \left(\frac{\partial\psi}{\partial z}\right)^2\right)e^{-2(\gamma-\psi)} \\
&+ \left(\frac{\partial^2(\gamma-\psi)}{\partial t^2} + \left(\frac{\partial(\gamma-\psi)}{\partial t}\right)^2 - \frac{\partial(\gamma-\psi)}{\partial t}\frac{\partial\psi}{\partial t}\right)e^{-2\psi}\,, \\
R^0{}_{212} = &\left(\frac{\partial^2(\gamma-\psi)}{\partial t\partial\rho} - \frac{\partial(\gamma-\psi)}{\partial t}\frac{\partial\psi}{\partial\rho}\right)e^{-\gamma}\,, \\
R^0{}_{303} = &\left(\left(\frac{\partial\psi}{\partial z}\right)^2 - \left(\frac{1}{\rho} - \frac{\partial\psi}{\partial\rho}\right)\frac{\partial\psi}{\partial\rho}\right)e^{-2(\gamma-\psi)} + \left(2\left(\frac{\partial\psi}{\partial t}\right)^2 - \frac{\partial^2\psi}{\partial t^2}\right)e^{-2\psi}\,, \\
R^0{}_{313} = &\left(\frac{\partial\psi}{\partial t}\frac{\partial\psi}{\partial\rho} - \frac{\partial^2\psi}{\partial t\partial\rho} - \left(\frac{1}{\rho} - \frac{\partial\psi}{\partial\rho}\right)\frac{\partial\gamma}{\partial t}\right)e^{-\gamma}\,, \\
R^0{}_{323} = &\left(\frac{\partial\psi}{\partial t}\frac{\partial\psi}{\partial z} - \frac{\partial^2\psi}{\partial t\partial z} + \frac{\partial\gamma}{\partial t}\frac{\partial\psi}{\partial z}\right)e^{-\gamma}\,,
\end{aligned}
$$

$$R^1{}_{201} = \left(\frac{\partial^2(\gamma-\psi)}{\partial t\partial z} - \frac{\partial(\gamma-\psi)}{\partial t}\frac{\partial\psi}{\partial z}\right)e^{-\gamma}\,,$$
$$R^1{}_{202} = \left(\frac{\partial(\gamma-\psi)}{\partial t}\frac{\partial\psi}{\partial\rho} - \frac{\partial^2(\gamma-\psi)}{\partial t\partial\rho}\right)e^{-\gamma}\,,$$
$$R^1{}_{212} = \left(-\frac{\partial^2(\gamma-\psi)}{\partial\rho^2} - \frac{\partial^2(\gamma-\psi)}{\partial z^2}\right)e^{-2(\gamma-\psi)} + \left(\frac{\partial(\gamma-\psi)}{\partial t}\right)^2 e^{-2\psi}\,,$$
$$R^1{}_{303} = \left(\frac{\partial^2\psi}{\partial t\partial\rho} + \left(\frac{1}{\rho} - \frac{\partial\psi}{\partial\rho}\right)\frac{\partial\gamma}{\partial t} - \frac{\partial\psi}{\partial t}\frac{\partial\psi}{\partial\rho}\right)e^{-\gamma}\,,$$
$$R^1{}_{313} = \left(\frac{\partial^2\psi}{\partial\rho^2} - \frac{\partial\gamma}{\partial\rho}\frac{\partial\psi}{\partial\rho} + \frac{1}{\rho}\frac{\partial\gamma}{\partial\rho} + \frac{1}{\rho}\frac{\partial\psi}{\partial\rho} + \frac{\partial(\gamma-\psi)}{\partial z}\frac{\partial\psi}{\partial z}\right)e^{-2(\gamma-\psi)}$$
$$\qquad - \left(\frac{\partial(\gamma-\psi)}{\partial t}\frac{\partial\psi}{\partial t}\right)e^{-2\psi}\,,$$
$$R^1{}_{323} = \left(\frac{\partial^2\psi}{\partial\rho\partial z} + \left(\frac{1}{\rho} - \frac{\partial\psi}{\partial\rho}\right)\frac{\partial\gamma}{\partial z} - \frac{\partial(\gamma-\psi)}{\partial\rho}\frac{\partial\psi}{\partial z}\right)e^{-2(\gamma-\psi)}\,,$$
$$R^2{}_{303} = \left(\frac{\partial^2\psi}{\partial t\partial z} - \frac{\partial\gamma}{\partial t}\frac{\partial\psi}{\partial z} - \frac{\partial\psi}{\partial t}\frac{\partial\psi}{\partial z}\right)e^{-\gamma}\,,$$
$$R^2{}_{313} = \left(\frac{\partial^2\psi}{\partial\rho\partial z} - \frac{\partial(\gamma-\psi)}{\partial\rho}\frac{\partial\psi}{\partial z} + \left(\frac{1}{\rho} - \frac{\partial\psi}{\partial\rho}\right)\frac{\partial\gamma}{\partial z}\right)e^{-2(\gamma-\psi)}\,,$$
$$R^2{}_{323} = \left(\frac{\partial^2\psi}{\partial z^2} - \frac{\partial\gamma}{\partial z}\frac{\partial\psi}{\partial z} - \left(\frac{1}{\rho} - \frac{\partial\psi}{\partial\rho}\right)\frac{\partial(\gamma-\psi)}{\partial\rho}\right)e^{-2(\gamma-\psi)} + \left(-\frac{\partial(\gamma-\psi)}{\partial t}\frac{\partial\psi}{\partial t}\right)e^{-2\psi}\,.$$

The remaining components vanish or can easily be obtained via the anti-symmetry of the Riemann tensor in its first and second index pair.

Ricci tensor and Ricci scalar

The Ricci tensor is defined by

$$R_{ij} = R^k{}_{ikj}\,.$$

Hence we get

$$R_{00} = \left(\frac{\partial^2\psi}{\partial\rho^2} + \frac{\partial^2\psi}{\partial z^2} + \frac{1}{\rho}\frac{\partial\psi}{\partial\rho}\right)e^{-2(\gamma-\psi)}$$
$$\qquad + 2\left(\frac{\partial(\gamma-\psi)}{\partial t}\frac{\partial\psi}{\partial t} - \frac{\partial^2(\gamma-\psi)}{\partial t^2} - \left(\frac{\partial(\gamma-\psi)}{\partial t}\right)^2 + \frac{1}{2}\frac{\partial^2\psi}{\partial t^2} - \left(\frac{\partial\psi}{\partial t}\right)^2\right)e^{-2\psi}\,,$$
$$R_{01} = \left(\frac{\partial^2\psi}{\partial t\partial\rho} + \frac{1}{\rho}\frac{\partial\gamma}{\partial t} - 2\frac{\partial\psi}{\partial t}\frac{\partial\psi}{\partial\rho} - \frac{\partial^2(\gamma-\psi)}{\partial t\partial\rho}\right)e^{-\gamma}\,,$$
$$R_{02} = \left(\frac{\partial^2\psi}{\partial t\partial z} - 2\frac{\partial\psi}{\partial t}\frac{\partial\psi}{\partial z} - \frac{\partial^2(\gamma-\psi)}{\partial t\partial z}\right)e^{-\gamma}\,,$$
$$R_{11} = \left(\frac{1}{\rho}\frac{\partial\gamma}{\partial\rho} + \frac{1}{\rho}\frac{\partial\psi}{\partial\rho} - \frac{\partial^2(\gamma-\psi)}{\partial\rho^2} - \frac{\partial^2(\gamma-\psi)}{\partial z^2} - 2\left(\frac{\partial\psi}{\partial\rho}\right)^2\right)e^{-2(\gamma-\psi)}$$
$$\qquad + 2\left(\frac{1}{2}\frac{\partial^2(\gamma-\psi)}{\partial t^2} - \frac{\partial(\gamma-\psi)}{\partial t}\frac{\partial\psi}{\partial t} + \left(\frac{\partial(\gamma-\psi)}{\partial t}\right)^2\right)e^{-2\psi}\,,$$

$$
\begin{aligned}
R_{12} &= \left(\frac{1}{\rho}\frac{\partial\gamma}{\partial z} - 2\frac{\partial\psi}{\partial\rho}\frac{\partial\psi}{\partial z}\right) e^{-2(\gamma-\psi)}\,,\\
R_{22} &= \left(-\frac{\partial^2(\gamma-\psi)}{\partial\rho^2} - \frac{\partial^2(\gamma-\psi)}{\partial z^2} - 2\left(\frac{\partial\psi}{\partial z}\right)^2 - \frac{1}{\rho}\frac{\partial(\gamma-\psi)}{\partial\rho}\right) e^{-2(\gamma-\psi)}\\
&\quad + 2\left(\frac{1}{2}\frac{\partial^2(\gamma-\psi)}{\partial t^2} - \frac{\partial(\gamma-\psi)}{\partial t}\frac{\partial\psi}{\partial t} + \left(\frac{\partial(\gamma-\psi)}{\partial t}\right)^2\right) e^{-2\psi}\,,\\
R_{33} &= \left(\frac{\partial^2\psi}{\partial\rho^2} + \frac{\partial^2\psi}{\partial z^2} + \frac{1}{\rho}\frac{\partial\psi}{\partial\rho}\right) e^{-2(\gamma-\psi)}\\
&\quad + 2\left(\left(\frac{\partial\psi}{\partial t}\right)^2 - \frac{1}{2}\frac{\partial^2\psi}{\partial t^2} - \frac{\partial(\gamma-\psi)}{\partial t}\frac{\partial\psi}{\partial t}\right) e^{-2\psi}\,.
\end{aligned}
$$

The other components vanish or are obtained via $R_{ij} = R_{ji}$. For the Ricci scalar we obtain

$$
\begin{aligned}
R = \eta^{ij}R_{ij} &= R_{00} - R_{11} - R_{22} - R_{33}\\
&= 2\left(\left(\frac{\partial\psi}{\partial\rho}\right)^2 + \left(\frac{\partial\psi}{\partial z}\right)^2 + \frac{\partial^2(\gamma-\psi)}{\partial\rho^2} + \frac{\partial^2(\gamma-\psi)}{\partial z^2} - \frac{1}{\rho}\frac{\partial\psi}{\partial\rho}\right) e^{-2(\gamma-\psi)}\\
&\quad + 2\left(3\frac{\partial^2\psi}{\partial t^2} - 2\frac{\partial^2\gamma}{\partial t^2} - 9\left(\frac{\partial\psi}{\partial t}\right)^2 - 3\left(\frac{\partial\gamma}{\partial t}\right)^2 + 10\frac{\partial\gamma}{\partial t}\frac{\partial\psi}{\partial t}\right) e^{-2\psi}\,.
\end{aligned}
$$

A.1 The vacuum Einstein field equations and a time dependent solution

The Einstein field equation in vacuum are given by

$$
\begin{aligned}
R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} &= 0\\
\iff R_{\mu\nu} &= 0\,.
\end{aligned}
$$

The time-independent case

Assuming that the functions ψ and γ do not depend on the time-coordinate t, the field equations simplify drastically:

$$
\frac{\partial^2\psi}{\partial z^2} + \frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial\psi}{\partial\rho}\right) = 0\,, \tag{48}
$$

$$
\frac{\partial\gamma}{\partial z} = 2\rho\frac{\partial\psi}{\partial\rho}\frac{\partial\psi}{\partial z}\,, \quad \frac{\partial\gamma}{\partial\rho} = \rho\left(\left(\frac{\partial\psi}{\partial\rho}\right)^2 - \left(\frac{\partial\psi}{\partial z}\right)^2\right)\,.
$$

The equation (48) is linear and for a given ψ the system for γ is linear, too. Furthermore the first equation is the integrability condition for the two other equations.

The time-dependent case

In the more general case, we get a more complex system of non-linear, partial differential equations of second order with three independent variables. From $R_{ij} = 0$ we obtain

$$\frac{\partial\gamma}{\partial z} = 2\rho\frac{\partial\psi}{\partial\rho}\frac{\partial\psi}{\partial z}\,, \quad \frac{\partial\gamma}{\partial\rho} = \rho\left(\left(\frac{\partial\psi}{\partial\rho}\right)^2 - \left(\frac{\partial\psi}{\partial z}\right)^2\right)$$

$$\frac{\partial\gamma}{\partial t} = 2\frac{\partial\psi}{\partial t} - \frac{1}{2\frac{\partial\psi}{\partial t}}\frac{\partial^2\psi}{\partial t^2}\,, \tag{49}$$

$$\frac{\partial\gamma}{\partial t} = 2\rho\left(\frac{\partial\psi}{\partial t}\frac{\partial\psi}{\partial\rho} + \rho\frac{\partial\psi}{\partial\rho}\frac{\partial^2\psi}{\partial t\partial\rho} - \frac{\partial^2\psi}{\partial t\partial\rho} - \rho\frac{\partial\psi}{\partial z}\frac{\partial^2\psi}{\partial t\partial z}\right)\,, \tag{50}$$

$$\frac{\partial^2\gamma}{\partial t^2} = 2\left(\frac{\partial^2\psi}{\partial t^2} - \left(\frac{\partial\psi}{\partial t}\right)^2\right)\,, \tag{51}$$

$$0 = 3\frac{\partial^2\gamma}{\partial t^2} - 4\frac{\partial^2\psi}{\partial t^2} - 12\frac{\partial\gamma}{\partial t}\frac{\partial\psi}{\partial t} + 10\left(\frac{\partial\psi}{\partial t}\right)^2 + 4\left(\frac{\partial\gamma}{\partial t}\right)^2\,, \tag{52}$$

$$0 = \frac{\partial}{\partial t}\left(\left(\frac{1}{\rho} - \frac{\partial\psi}{\partial\rho}\right)\frac{\partial\psi}{\partial z}\right) - \frac{1}{\rho}\frac{\partial\psi}{\partial t}\frac{\partial\psi}{\partial z}\,, \tag{53}$$

$$0 = \frac{\partial^2\psi}{\partial z^2} + \frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial\psi}{\partial\rho}\right) \quad \text{(integrability condition for } z \leftrightarrow \rho\text{)}\,. \tag{54}$$

The equations we found in the the time-independent case are not changed. We used the integrability condition (54) to derive the other equations. Two further integrability conditions emerge:

$$\begin{aligned}0 = &\frac{\partial\psi}{\partial\rho}\left(\frac{\partial\psi}{\partial t}\right)^2\frac{\partial^2\psi}{\partial t\partial\rho} - \frac{\partial\psi}{\partial z}\left(\frac{\partial\psi}{\partial t}\right)^2\frac{\partial^2\psi}{\partial t\partial z} - \frac{1}{\rho}\left(\frac{\partial\psi}{\partial t}\right)^2\frac{\partial^2\psi}{\partial t\partial\rho}\\ &- \frac{1}{4\rho}\frac{\partial^2\psi}{\partial t\partial\rho}\frac{\partial^2\psi}{\partial t^2} + \frac{1}{4\rho}\frac{\partial\psi}{\partial t}\frac{\partial^3\psi}{\partial t^2\partial\rho}\,,\\ 0 = &\frac{\partial\psi}{\partial\rho}\left(\frac{\partial\psi}{\partial t}\right)^2\frac{\partial^2\psi}{\partial t\partial z} + \frac{\partial\psi}{\partial z}\left(\frac{\partial\psi}{\partial t}\right)^2\frac{\partial^2\psi}{\partial t\partial\rho} - \frac{1}{\rho}\left(\frac{\partial\psi}{\partial t}\right)^2\frac{\partial^2\psi}{\partial t\partial z}\\ &- \frac{1}{4\rho}\frac{\partial^2\psi}{\partial t\partial z}\frac{\partial^2\psi}{\partial t^2} + \frac{1}{4\rho}\frac{\partial\psi}{\partial t}\frac{\partial^3\psi}{\partial t^2\partial z}\,.\end{aligned}$$

Just as in the static case, we want to write this system of equations in a way that we get three equations for γ, which determine this function for a given ψ.

The other differential equations determine the function ψ.

$$\frac{\partial\gamma}{\partial z} = 2\rho\frac{\partial\psi}{\partial\rho}\frac{\partial\psi}{\partial z}\,, \quad \frac{\partial\gamma}{\partial\rho} = \rho\left(\left(\frac{\partial\psi}{\partial\rho}\right)^2 - \left(\frac{\partial\psi}{\partial z}\right)^2\right)\,, \quad \frac{\partial\gamma}{\partial t} = 2\frac{\partial\psi}{\partial t} - \frac{1}{2\frac{\partial\psi}{\partial t}}\frac{\partial^2\psi}{\partial t^2}\,,$$

$$0 = \frac{\partial}{\partial t}\left(\left(\frac{1}{\rho} - \frac{\partial\psi}{\partial\rho}\right)\frac{\partial\psi}{\partial z}\right) - \frac{1}{\rho}\frac{\partial\psi}{\partial t}\frac{\partial\psi}{\partial z}\,, \tag{55}$$

$$0 = \frac{\partial^2\psi}{\partial z^2} + \frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial\psi}{\partial\rho}\right) \quad \text{(integrability condition for } z \leftrightarrow \rho)\,, \tag{56}$$

$$\begin{aligned} 0 = {} & \rho\left(\frac{\partial\psi}{\partial t}\right)^2\frac{\partial\psi}{\partial\rho} + \rho^2\frac{\partial\psi}{\partial\rho}\frac{\partial\psi}{\partial t}\frac{\partial^2\psi}{\partial t\partial\rho} - \rho\frac{\partial\psi}{\partial t}\frac{\partial^2\psi}{\partial t\partial\rho} \\ & - \rho^2\frac{\partial\psi}{\partial z}\frac{\partial\psi}{\partial t}\frac{\partial^2\psi}{\partial t\partial z} - \left(\frac{\partial\psi}{\partial t}\right)^2 + \frac{1}{4}\frac{\partial^2\psi}{\partial t^2}\,, \end{aligned} \tag{57}$$

$$0 = \left(\frac{\partial^2\psi}{\partial t^2}\right)^2 - 4\left(\frac{\partial\psi}{\partial t}\right)^4\,, \tag{58}$$

$$0 = 2\left(\frac{\partial^2\psi}{\partial t^2}\right)^2 - \frac{\partial\psi}{\partial t}\frac{\partial^3\psi}{\partial t^3}\,, \tag{59}$$

$$\begin{aligned} 0 = {} & \frac{\partial\psi}{\partial\rho}\left(\frac{\partial\psi}{\partial t}\right)^2\frac{\partial^2\psi}{\partial t\partial\rho} - \frac{\partial\psi}{\partial z}\left(\frac{\partial\psi}{\partial t}\right)^2\frac{\partial^2\psi}{\partial t\partial z} - \frac{1}{\rho}\left(\frac{\partial\psi}{\partial t}\right)^2\frac{\partial^2\psi}{\partial t\partial\rho} \\ & - \frac{1}{4\rho}\frac{\partial^2\psi}{\partial t\partial\rho}\frac{\partial^2\psi}{\partial t^2} + \frac{1}{4\rho}\frac{\partial\psi}{\partial t}\frac{\partial^3\psi}{\partial t^2\partial\rho} \\ & \text{(integrability condition for } t \leftrightarrow \rho)\,, \end{aligned} \tag{60}$$

$$\begin{aligned} 0 = {} & \frac{\partial\psi}{\partial\rho}\left(\frac{\partial\psi}{\partial t}\right)^2\frac{\partial^2\psi}{\partial t\partial z} + \frac{\partial\psi}{\partial z}\left(\frac{\partial\psi}{\partial t}\right)^2\frac{\partial^2\psi}{\partial t\partial\rho} - \frac{1}{\rho}\left(\frac{\partial\psi}{\partial t}\right)^2\frac{\partial^2\psi}{\partial t\partial z} \\ & - \frac{1}{4\rho}\frac{\partial^2\psi}{\partial t\partial z}\frac{\partial^2\psi}{\partial t^2} + \frac{1}{4\rho}\frac{\partial\psi}{\partial t}\frac{\partial^3\psi}{\partial t^2\partial z} \\ & \text{(integrability condition for } t \leftrightarrow z)\,. \end{aligned} \tag{61}$$

The equations (55) and (56) are not changed. By setting (49) and (50) equal we get (57). Equation (58) is the result of the substitution of (49) and (51) into (52). Equation (59) is the result of setting equal the time derivative of (49) with (51) where we added (58).

The differential equation (58) is easily solved and determines the time-dependence of ψ:

$$\psi(t, \rho, z) = \pm\frac{1}{2}\ln\left(t - t_0(\rho, z)\right) + A(\rho, z)$$

This is compatible with (51), (52) and (59). t_0 and A are the unknown z- and ρ-dependent parts of ψ. Substituting this result into (56), we get two equations for t_0:

$$\frac{\partial t_0}{\partial z} = \pm\frac{\partial t_0}{\partial\rho}\,, \quad \frac{\partial^2 t_0}{\partial z^2} + \frac{\partial^2 t_0}{\partial\rho^2} + \frac{1}{\rho}\frac{\partial t_0}{\partial\rho} = 0\,.$$

The first equation indicates that t_0 has to be of the form $t_0 = t_0(z \pm \rho)$, hence the second equation yields

$$2t_0'' \pm \frac{1}{\rho} t_0' = 0 \Rightarrow t_0' = 0 \Rightarrow t_0 = \text{const.}.$$

t_0 has to be constant. With this result we look at the equations (55) - (57).

$$\begin{aligned} \frac{\partial A}{\partial z} &= 0\,, \\ 2 \pm 1 - 2\rho \frac{\partial A}{\partial \rho} &= 0\,, \\ \frac{\partial^2 A}{\partial z^2} + \frac{\partial^2 A}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial A}{\partial \rho} &= 0\,. \end{aligned}$$

Hence ψ and γ have cylindrical symmetry. As a result we obtain the solution

$$\begin{aligned} \psi(t,\rho) &= \ln\left(\sqrt{\frac{\rho}{t}}\right)\,, \\ \gamma(t,\rho) &= \frac{1}{2}\ln\left(\frac{\sqrt{\rho}}{t}\right)\,, \end{aligned}$$

and the corresponding metric

$$(g_{\mu\nu}) = \begin{pmatrix} \frac{\rho}{t} & 0 & 0 & 0 \\ 0 & -\frac{1}{\sqrt{\rho}} & 0 & 0 \\ 0 & 0 & -\frac{1}{\sqrt{\rho}} & 0 \\ 0 & 0 & 0 & -\rho t \end{pmatrix}.$$

Here we chose the integration constants and t_0 to be zero. This solution has been checked with MATHEMATICA. The Ricci tensor vanishes as expected.

This solution requires further investigation. It is cylindrically symmetry. The invariant

$$R_{\mu\nu\lambda\rho}R^{\mu\nu\lambda\rho} = \frac{3}{4\rho^3}\,,$$

which does not depend on t, exhibits a singularity at $\rho = 0$.

B Legendre-functions of the second kind

The Legendre-differential equation

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[(1-x^2)\frac{\mathrm{d}f(x)}{\mathrm{d}x}\right]+l(l+1)f(x)=0 \tag{62}$$

is not only solved by the Legendre-polynomials $P_l(x)$ for $l \in \mathbb{Z}$, where $-\infty < x < \infty$ and x can be continued to the complex plane. There has to be a second independent solution $Q_n(x)$.

The first three solutions of these *Legendre-functions of the second kind* are

$$\begin{aligned} Q_0(z) &= \frac{1}{2}\ln\left(\frac{z+1}{z-1}\right), \quad Q_1(z)=\frac{z}{2}\ln\left(\frac{z+1}{z-1}\right)-1\,, \\ Q_2(z) &= \frac{1}{2}P_2(z)\ln\left(\frac{z+1}{z-1}\right)-\frac{3}{2}P_2(z)\,, \end{aligned}$$

where $z \in \mathbb{C} \setminus [-1,1]$. The usual cut in the definition of the logarithm corresponds to the interval $[-1,1]$. One cannot continue the function analytically through this interval. In our case of the coordinate λ in 2.2 this is not necessary, because $\lambda \geq 1$ always holds.

But it is possible to define values on this interval by

$$Q_0(x) := \frac{1}{2}\lim_{\epsilon\to 0}(Q_0(x+i\epsilon)+Q_0(x-i\epsilon))\,.$$

Now $x \in (-1,1)$ holds. We look at $Q_0(x)$ as an example:

$$\begin{aligned} Q_0(x) &= \frac{1}{4}\lim_{\epsilon\to 0}\left[\ln\left(\frac{x+1+i\epsilon}{x-1+i\epsilon}\right)+\ln\left(\frac{x+1-i\epsilon}{x-1-i\epsilon}\right)\right]=\frac{1}{4}\lim_{\epsilon\to 0}\left(\frac{(x+1)^2+\epsilon^2}{(x-1)^2+\epsilon^2}\right) \\ &= \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right) \end{aligned}$$

The resulting function on $\mathbb{C} \setminus \{-1,1\}$ is not continuous, but it solves (62) for $x \in (-1,1)$.

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Erklärung nach §13(8) der Prüfungsordnung für den Bachelor-Studiengang Physik und den Master-Studiengang Physik an der Universität Göttingen

Hiermit erkläre ich, dass ich diese Abschlussarbeit selbstständig verfasst habe, keine anderen als die angegebenen Quellen und Hilfsmittel benutzt habe und alle Stellen, die wörtlich oder sinngemäß aus veröffentlichten Schriften entnommen wurden, als solche kenntlich gemacht habe.

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Göttingen, den 24. Juli 2011
(Timon Emken)